Aspiration and Confidence under Ambiguity

Jingyi Xue[*†]

April 3, 2013

Abstract

This paper develops a model of uncertainty in which a decision maker evaluates an act based on his aspiration and his confidence in this aspiration. Each act corresponds to a trade-off line between the two criteria: The more he aspires, the less his confidence in achieving the aspiration level. The decision maker ranks an act by the optimal combination of aspiration and confidence on its trade-off line according to an aggregating preference of his over the two-criteria plane. To reveal the decision maker's perception about uncertainty, this paper introduces confidence orders in addition to preference orders; the confidence orders compare the decision maker's confidence in all aspiration levels of all acts. Axioms are imposed on both confidence and preference orders, which yields a capacity over all priors to represent the confidence order, and the above decision rule to represent the preference order. The aggregating preference over the aspiration and confidence criteria plane is endogenously determined.

1 Introduction

The fundamental work of Savage (1954) establishes a beautiful theory to derive the subjective probability distribution from a decision maker's preferences.[1] For example, if the decision maker prefers to bet on event A than event B, then A is revealed to be more likely

[*]I am indebted to Simon Grant, Siyang Xiong, Hervé Moulin, In-Koo Cho, Edi Karni, Peter Klibanoff, Mark Machina, Francois Maniquet, Jianrong Tian and Stephen Wolff for invaluable discussion and suggestions. I have special thanks to Jean-Francois Mertens. All errors are my own.

[†]Department of Economics, Rice University, jx2@rice.edu

[1]The early idea goes back to Ramsey (1931) and de Finetti (1937).

for him than B. This approach is challenged by Ellsberg (1961)'s famous thought experiments. One example of Ellsberg (1961) is to consider an urn with 90 balls in it, 30 red and 60 either black or green. A ball is drawn from this urn. You are asked first to choose from betting on red and betting on green. Next, you are asked to choose from betting on "either red or green" and "either black or green". For each draw, the payoff structure is the same: you get 100 dollars if your bet is right and nothing if it is wrong. As Ellsberg (1961) suggests and many later experiments confirm, most decision makers prefer to bet on red in the first case and "either black or green" in the second. If a decision maker does have a subjective belief, then his first choice reveals that the probability of the ball being green is less than $\frac{1}{3}$, while the second choice reveals it more than $\frac{1}{3}$.

This example shows that subjective probability may not exist. In this decision problem, there exists ambiguity, and ambiguity matters for a decision maker's choices. Indeed, the proportion of green balls is *unknown* and the decision maker has a aversion to betting on the ambiguous events. The literature goes further by asking mainly two questions: (1) Can we distinguish between ambiguous and unambiguous events, and derive a decision maker's subjective probability on unambiguous events from his preference? (2) How does a decision maker make choices under ambiguity?

In responding to the first question, most studies define an ambiguous event via the "inconsistency" or reversal of preferences in some fashion, and define unambiguous events and acts accordingly.[2] Axioms are imposed on preferences to deliver a subjective probability over the unambiguous events and also probabilistic sophistication behavior on unambiguous acts. Many criticisms arise when researchers find examples[3] where some obviously ambiguous events are identified as unambiguous and some apparently unambiguous events are identified as ambiguous according to the definitions. One important reason for such misidentification proposed by Klibanoff, Marinacci and Mukerji (2011) is that some definition confounds ambiguity with ambiguity attitude. The inconsistency of preference may result from a decision maker's changing ambiguity attitude rather than the ambiguity of some event. Nehring (2006) finds that even the consistency of preference may have nothing to do with the existence of a subjective probability on relevant events.

In responding to the second question, many models have proposed different axioms on

[2]E.g., see Nehring (1999), Epstein and Zhang (2001), Ghirardato and Marinacci (2001), Zhang (2002), Klibanoff, Marinacci and Mukerji (2005).

[3]E.g., see Klibanoff, Marinacci and Mukerji (2011), Nehring (2006).

preferences to characterize different decision rules.[4] Implicitly or explicitly, most decision rules base the interpretation on two factors: the ambiguity that a decision maker feels, and his attitude toward this ambiguity. For example, Gilboa and Schmeidler (1989)'s "maxmin expected utility" (MEU) models the ambiguity as a set of priors and presents a decision maker's ambiguity attitude by his taking the minimum expected utilities over these priors. While ambiguity reflects a decision maker's personal perception of the underlying situation, ambiguity attitude shows how he responds to it. His preference is determined by both factors, but they are subjective and unobservable for the outside modeler. Hence, with only the choice data, there is some arbitrariness in explaining a decision maker's behavior by these two free variables. For instance, in contrast to the MEU model, the decision maker may in reality have a larger set of priors and evaluate an act by a weighted average of the minimum and maximum expected utility over this set of priors, rather than always considering the worst scenario. Although Ghirardato, Maccheroni and Marinacci (2004) characterize the α-MEU representation[5] and claim that their model completely separates ambiguity and ambiguity attitude, Eichberger, Grant, Kelsey and Koshevoy (2011) shows that their identification of the set of priors summarizing a decision maker's perceived ambiguity fails for general α-MEU preferences with $\alpha \neq 0$ or 1.

The difficulty of answering both questions is same. Can we identify a decision maker's perceived ambiguity and his ambiguity attitude with only the data of choices? Nehring (2006) conjectures that the purely behavioral approach itself may have deep-seated limitation in identifying subjective beliefs under ambiguity. Klibanoff, Marinacci and Mukerji (2005) represent a decision maker's perception about ambiguity as a probability over priors, and resort to the "second-order" acts, i.e., the bets on the priors, to reveal his belief over priors. They notice that there is a question whether the preference with respect to the second-order acts are observable: "When it is not evident we may need something richer than behavioral data, perhaps cognitive data or thought experiments, to help us reveal the decision maker's belief over Δ [the priors]."

[4] E.g., see Gilboa and Schmeidler (1989), Schmeidler (1989), Klibanoff, Marinacci and Mukerji (2005), Marinacci, Maccheroni and Rustichini (2006), Strzalecki (2011).

[5] The functional form is

$$V(f) = \alpha \min_{C} E_p u(f) + (1-\alpha) \max_{C} E_p u(f)$$

where f is an act, $u(f)$ is the state-contingent utility profile, $E_p u(f)$ is the expectation of $u(f)$ with respect to a prior p, C is a non-empty closed convex set of priors, and $\alpha \in [0, 1]$ is a weight parameter.

This paper achieves the goal of identifying a decision maker's perceived ambiguity by explicitly introducing a *confidence order* in addition to the preference order. The confidence order can be obtained from psychological data. It ranks the degree of a decision maker's confidence in aspiring a particular return from an act. More precisely, an act generates different expected payoffs under different priors. Given an act and an aspired expected payoff, if the expected payoff of the act under a prior is no less than the aspiration level, then the prior is called a *supportive* prior. The higher the aspiration level, the less the supportive priors, and the lower the confidence in achieving the aspiration. The confidence depends on the decision maker's belief about the likelihood of the set of supportive priors. It is similar to the second-order belief over priors proposed by Klibanoff, Marinacci and Mukerji (2005). However, under the axioms imposed on this confidence order, a capacity instead of a probability over the priors is elicited. Capacity is a generalization of probability and is accepted in the literature as a more relevant notion to model the belief under ambiguity.[6]

Together with the confidence order, this paper axiomatizes a class of preferences that base the evaluation of acts on two criteria: a decision maker's aspiration from the act, and his confidence in achieving this aspiration level. As discussed above, the confidence in aspiration decreases in the aspiration level. Each act corresponds to a trade-off between aspiration and confidence. The preferences that we characterize evaluate an act by the optimal combination of aspiration and confidence according to another aggregating preference over the two-criteria plane. The aggregating preference is endogenously determined from the preference over acts.

Although several papers compare the degrees of decision makers' ambiguity aversion based only on their preferences, such comparison is meaningful only when the decision makers perceive the same ambiguity; otherwise, ambiguity and ambiguity attitude will be confounded. In this paper, since decision makers' perception about ambiguity is revealed by confidence orders, the comparison of ambiguity attitude can be obtained. A decision maker is more ambiguity averse than another if (1) he perceives the same ambiguity as the other, that is, they have the identical confidence order, and (2) at each aspiration level, according to the aggregating preference, he requires more degree of confidence to achieve the same level of satisfaction, that is, the indifference curves of his aggregating preference lie above those of the other.

An important feature of this class of preferences is that it allows a decision maker's

[6]See eg. Schmeidler (1989).

ambiguity attitude to change across acts. Most current literature assumes to some extent that a decision maker always display the same degree of ambiguity aversion.

2 Setup

We denote by $\mathbb{R}$ the set of all reals, and $\mathbb{Z}_+$ the set of positive integers. Let S be a finite set of *states of the world*. Suppose that $|S| \geq 2$ where $|\cdot|$ denotes the cardinality of a set. A subset of S is called an *event*. Let $\Delta(S)$ be the set of all probability measures on S. We identify $\Delta(S)$ with the standard $|S|-1$ dimensional simplex in $\mathbb{R}^{|S|}$, i.e., the set $\{(p_1, ..., p_{|S|}) \in \mathbb{R}^{|S|} \mid \sum_{i=1}^{|S|} p_i = 1 \text{ and } p_i \geq 0 \text{ for all } i\}$.

Let X be a set of *outcomes*. Suppose that X is a connected metric space. Let $\Delta(X)$ be the set of all Borel probability measures on X. An element $l \in \Delta(X)$ is called a *lottery* on X. For all $l \in \Delta(X)$ and $x \in X$, we write $l(x)$ to denote $l(\{x\})$. A lottery $l \in \Delta(X)$ is a *simple lottery* if $l(x) > 0$ for finitely many $x \in X$, and $\sum_{x \in X} l(x) = 1$. Let $\mathcal{L}_0$ be the set of all simple lotteries. For all $l \in \mathcal{L}_0$, denote by supp(l) the support of l, i.e. the set $\{x \in X \mid l(x) > 0\}$.

Let $f : S \to X$ be a (Savage) *act* which specifies an outcome in each state. Let $\mathcal{F}_0 = X^S$ be the set of all the (Savage) acts. Consider the set $\mathcal{F}_1 = \mathcal{F}_0 \times \Delta(S)$. A pair $(f, p) \in \mathcal{F}_1$ is an *informational act* which denotes an act f with the information that the probability over S is p. This prior information can be objectively given. For example, consider an Ellsberg's urn containing black and white balls with *known* proportion. Betting on the color of a ball drawn from this urn is an act with given probabilistic information. Alternatively, we may also think of it as an act with postulated prior in the decision maker's mind. For example, consider two prospects f and g such that f yields \$0 in the bad economic condition and \$100 in the good economic condition, while g produces \$10 and \$60 in respective economic conditions. A decision maker may compare his welfare in the following two situations as a thought expriment: First, he owns the prospect f while the probability of good economic condition is 80%; second, he owns the prospect g while the corresponding probability is 50%. Such comparison depends not only on acts but also on associated probabilities.

Each pair of an act and a prior generates a lottery in the natural way. More precisely, given $(f, p) \in \mathcal{F}_1$, if $l \in \Delta(X)$ and $l(x) = \sum_{f(s)=x} p_s$ for all $x \in X$, then l is said to be *generated* by (f, p). Let $\mathcal{L}_1$ be the set of all lotteries generated by some $(f, p) \in \mathcal{F}_1$. Endow $\mathcal{L}_1$ with the weak topology induced by the collection of real-valued functions on $\mathcal{L}_1$ of the form $\int \eta dl$ for all $l \in \mathcal{L}_1$, where η is a continuous real-valued function on X. Note that with this topology, a sequence $\{l_n\}_{n=1}^{\infty}$ of elements in $\mathcal{L}_1$ converges to $l \in \mathcal{L}_1$ if and only if $\lim_{n\to\infty} \int \eta dl_n = \int \eta dl$ for every real-valued continous function η on X.

Given $l_1, l_2 \in \mathcal{L}_0$ and $\lambda \in [0,1]$, we define their convex combination $\lambda l_1 + (1-\lambda)l_2$ as a lottery in $\mathcal{L}_0$ such that $[\lambda l_1 + (1-\lambda)l_2](Y) = \lambda l_1(Y) + (1-\lambda)l_2(Y)$ for all Borel sets $Y \subseteq X$. For simplicity we write $l_1\lambda l_2$ instead of $\lambda l_1 + (1-\lambda)l_2$. Note that $\mathcal{L}_0$ is closed under convex combination, but $\mathcal{L}_1$ is not. Given $l_1, l_2 \in \mathcal{L}_0$ and $\lambda \in (0,1)$, $l_1\lambda l_2 \in \mathcal{L}_1$ if and only if $|\text{supp}(l_1\lambda l_2)| \leq |S|$. More precisely, $\mathcal{L}_1$ is the set of all simple lotteries which assign positive probability to at most $|S|$ elements in X.

We model a decision maker's preference $\succsim$ as a binary relation on $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$. Let $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$ as usual. Given $x \in X$, let f_x denote the *constant act* $f \in \mathcal{F}_0$ such that $f(s) = x$ for all $s \in S$, and l_x the *degenerate lottery* $l \in \mathcal{L}_1$ such that $l(x) = 1$. Given $x, y \in X$ and $A \subseteq S$, let xAy denote an act in $\mathcal{F}_0$ such that $xAy(s) = x$ for all $s \in A$ and $xAy(s) = y$ for all $s \in \Delta(S) \setminus A$.

Given a set Z and an order $\trianglerighteq$ on Z, let $\max Z = \{z \in Z \mid z \trianglerighteq z' \text{ for all } z' \in Z\}$. We write $\max Z \trianglerighteq z$ if $z' \trianglerighteq z$ for all $z' \in \max Z$. Similarly, $\min Z$ and $z \trianglerighteq \min Z$ are defined. A real-valued function T on Z is said to represent $\trianglerighteq$ on Z if and only if $T(z) \geq T(z') \Leftrightarrow z \trianglerighteq z'$ for all $z, z' \in Z$.

Given $a, a' \in \mathbb{R}^N$, $N \in \mathbb{Z}_+$, we say $a \geq a'$ ($a > a'$) if $a_n \geq a'_n$ ($a_n > a'_n$) for all $n = 1, ..., N$. Let T be a real-valued function on $\mathbb{R}^N$, $N \in \mathbb{Z}_+$. We say T is increasing (decreasing) if $T(a) \geq T(a')$ ($T(a) \leq T(a')$) for all $a \geq a'$ in $\mathbb{R}^N$. We say T is strictly increasing (strictly decreasing) if T is increasing (decreasing) and $T(a) > T(a')$ ($T(a) < T(a')$) for all $a > a'$ in $\mathbb{R}^N$.

This setup differs from the two classic settings — that of Savage (1954) and of Anscome and Aumann (1963) — for preferences under uncertainty. Savage (1954) assumes that preferences are defined over functions from states to outcomes, i.e., acts in $\mathcal{F}_0$. These elements are considered as *subjective acts* since no probability information is given as prior. Anscome and Aumann (1963) additionally assume that there is an objective random device so that preferences are defined over acts which are functions from states to *objective lotteries*. The existence of objective lotteries largely simplifies analysis, but it is under the critics that there are situations which are lack of random device.

Our setup retains the Savage acts while introduces more objects of choice from $\mathcal{F}_1$. As discussed before, these alternatives can be viewed either as acts with objectively given prior information or acts with postulated priors in the decision maker's own thought experiment. By imposing a neutrality axiom (see Section 2), the informational acts in $\mathcal{F}_1$ are regarded as lotteries. Therefore, while Savage (1954) does not assume the existence of lotteries, and Anscombe and Aumann (1963) explicitly introduce lotteries as preliminaries, in our setting, lotteries are endogenously generated. Thus, this setup provides a natural environment which keeps the advantage of Anscome and Aumann (1963)'s setting without relying on random device.

3 Expected utility representation on informational acts

We consider the following axioms on $\succsim$ restricted to $\mathcal{F}_1$.

A.1. Weak Order. $\succsim$ is complete and transitive.

A.2.1. Neutrality. If $(f, p), (g, q) \in \mathcal{F}_1$ generate the same lottery, then $(f, p) \sim (g, q)$.

Each lottery in $\mathcal{L}_1$ corresponds to a set of informational acts which generate this lottery, and A.2.1 says that those informational acts are equivalent with respect to $\succsim$. In the following when we do not distinguish informational acts which are indifferent, we would interchangeably use $\mathcal{F}_1$ and $\mathcal{L}_1$ as needed for convenience. Since each informational act generate a lottery, it is natural to seek for the expected utility representation for preferences restricted to informational acts. However, as mentioned above, the set of generated lotteries is not a convex set, so the mixture space theorem (Herstein and Milnor (1953)) does not directly apply. Nevertheless, von Neumann-Morgenstern's expected utility representation can still be obtained on the set of simple lotteries which assign positive probability to at most $|S|$ outcomes.

A.3.1. Continuity. For any $l_1 \in \mathcal{L}_1$, the sets $\{l \in \mathcal{L}_1 \mid l \succsim l_1\}$ and $\{l \in \mathcal{L}_1 \mid l_1 \succsim l\}$ are closed in $\mathcal{L}_1$.

A.4.1. Independence. For any $l_1, l_2, l_3 \in \mathcal{L}_1$ and $\lambda \in (0, 1)$, $l_1 \succ l_2$ implies that $l_1 \lambda l_3 \succ l_2 \lambda l_3$ if they exist in $\mathcal{L}_1$.

A.5. Unboundedness. There exist l_x, l_y in $\mathcal{L}_1$ such that (1) $l_x \succ l_y$, and (2) for all $\lambda \in (0, 1)$ there are $z_1, z_2 \in X$ satisfying $l_y \succ l_{z_1} \lambda l_x$ and $l_{z_2} \lambda l_y \succ l_x$.

Let $u : X \to \mathbb{R}$ be a utility function of outcomes. Given $f \in \mathcal{F}$, let $u(f)$ denote a function in $\mathbb{R}^S$ assigning $u(f(s))$ to each $s \in S$. Thus $u(f)$ transfer each act f to a state-contingent utility function. Given $f \in \mathcal{F}_0$, $u : X \to \mathbb{R}$ and $p \in \Delta(S)$, let $E_p u(f)$ denote the expected value of $u(f)$ with respect to p, i.e., $E_p u(f) = \sum_{s \in S} u(f(s)) p_s$. Given $l \in \mathcal{L}_0$ and $u : X \to \mathbb{R}$, let $E_l u$ denote the expected value of u with respect to l, i.e., $E_l u = \sum_{x \in X} u(x) l(x)$. Note that if $(f, p), (g, q) \in \mathcal{F}_1$ generate the same lottery $l \in \mathcal{L}_1$, then $E_l u = E_p u(f) = E_q u(g)$.

Lemma 1. *Suppose that $\succsim$ satisfies A.1. Then $\succsim$ satisfies A.2.1, A.3.1 and A.4.1 if and only if there is a continuous function $u : X \to \mathbb{R}$ such that $(f, p) \succsim (g, q)$ in $\mathcal{F}_1 \Leftrightarrow E_p u(f) \geq E_q u(g)$. Moreover, u is unique up to a positive affine transformation. The set $u(X)$ is $\mathbb{R}$ if and only if $\succsim$ additionally satisfies A.5.*

4 Confidence order

Consider another binary relation $\succsim'$ on $\mathcal{M} = (\mathcal{F}_0 \times \mathcal{L}_1) \cup (\mathcal{L}_1 \times X)$, with $\succ'$ and $\sim'$ denoting its asymmetric and symmetric parts respectively. Given a lottery l, a decision maker may aspire to achieve an outcome at least as good as x from l. Similarly, given an act f, he may aspire it to be at least as good as some lottery l. Different aspiration levels give rise to different degrees of confidence. A higher aspiration level corresponds to a lower degree of confidence. A pair (f, l) (or (l, x)) stands for the degree of confidence that a decision maker has in aspiring an act f to be at least as good as a lottery l (or an outcome x).

By restricting $\succsim$ to degenerate lotteries in $\mathcal{L}_1$, we get an induced preference on X — $y \succsim x$ if and only if $l_y \succsim l_x$ for all $x, y \in X$. Given $l \in \mathcal{L}_1$, let $L_l : X \to [0, 1]$ denote the decumulative distribution function such that $L(x) = \sum_{y \succsim x} l(y)$ for all $x \in X$.

To reflect the nature of a confidence order, consider the axioms below.

A.1′ Order. $\succsim'$ is complete and transitive.

A.2′ Continuity. For any $x \in X$, $f \in \mathcal{F}_0$ and $l_1, l_2, l_3 \in \mathcal{L}_1$ such that $l_1 \lambda l_3 \in \mathcal{L}_1$ for some $\lambda \in (0, 1)$,

(1) if $(l_1, x) \succ' (f, l_2) \succ' (l_3, x)$, then there exist $\alpha, \beta \in (0, 1)$ such that $(l_1 \alpha l_3, x) \succ' (f, l_2) \succ' (l_1 \beta l_3, x)$;

(2) if $(f,l_1) \succ' (l_2,x) \succ' (f,l_3)$, then there exists $\alpha \in (0,1)$ such that $(f, l_1\alpha l_3) \succ' (l_2,x)$ when $(f,l_1) \notin \max \mathcal{M}$ or $(f,l_3) \notin \min \mathcal{M}$, and there always exists $\beta \in (0,1)$ such that $(l_2,x) \succ' (f, l_1\beta l_3)$.

A.3.1'. Monotonicity. Let $x_1, x_2 \in X$, $f_1, f_2 \in \mathcal{F}_0$ and $l_1, l_2 \in \mathcal{L}_1$ be given. Then

(1) $(l_1, x_1) \succsim' (l_2, x_2)$ if and only if $L_1(x_1) \geq L_2(x_2)$;

(2) if $(f_2, p) \succsim l_2$ implies $(f_1, p) \succsim l_1$ for each $p \in \Delta(S)$, then $(f_1, l_1) \succsim' (f_2, l_2)$.

A.4'. Neutrality. For any $x_1, x_2 \in X$, $(f_{x_1}, l_{x_2}) \sim' (l_{x_1}, x_2)$.

Given $(f,l) \in \mathcal{F}_0 \times \mathcal{L}_1$, let $D(f,l) = \{p \in \Delta(S) \mid (f,p) \succsim l\}$. Given $(f_1,l_1),(f_2,l_2) \in \mathcal{F}_0 \times \mathcal{L}_1$, $D(f_1,l_1) \supseteq D(f_2,l_2)$ means that there are more probabilities to sustain f_1 to achieve the aspiration level l_1 than f_2 to achieve l_2. In other words, the aspiration of l_1 from f_1 bears more tolerance to prior misidentification. Axiom 3.1' says that for an objective lottery l, the confidence of aspiring x comes from its likelihood to obtain x or better outcomes, while for a subjective act f, the confidence of aspiring l depends on his perception about how large is the set $D(f,l)$. In the latter case, since the confidence corresponds to the degree of tolerance to prior misidentification, we will call it the *robustness* of aspiring l from f.

Let $\mathcal{D} = \{D(f,l) \subseteq \Delta(S) \mid (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1\}$. Note that $\emptyset, \Delta(S) \in \mathcal{D}$. A *capacity*[7] on $\mathcal{D}$ is a function $c : \mathcal{D} \to \mathbb{R}$ such that $c(D_1) \geq c(D_2)$ if $D_1 \supseteq D_2$, and $c(\emptyset) = 0$, $c(\Delta(S)) = 1$. A capacity c on $\mathcal{D}$ is *upper continuous* if $\lim_{n\to\infty} c(D_n) = c(\cap_{n=1}^{\infty} D_n)$ for any non-increasing sequence $\{D_n\}_{n=1}^{\infty}$ of elements in $\mathcal{D}$ and $\cap_{n=1}^{\infty} D_n$ in $\mathcal{D}$. It is *lower continuous* if $\lim_{n\to\infty} c(D_n) = c(\cup_{n=1}^{\infty} D_n)$ for any non-decreasing sequence $\{D_n\}_{n=1}^{\infty}$ of elements in $\mathcal{D}$ and $\cup_{n=1}^{\infty} D_n$ in $\mathcal{D}$. A capacity c on $\mathcal{D}$ is *continuous* if it is both upper and lower continuous.

Lemma 2. *Suppose that $\succsim$ satisfies Axiom 1, 2.1 - 4.1, 5, and u is given as in Lemma 1. The following statements are equivalent.*

(1) $\succsim'$ satisfies Axiom 1', 2', 3.1', 4'.

(2) There exists a unique continuous capacity c on $\mathcal{D}$ such that

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l,x) \in \mathcal{L}_1 \times X \\ c(D(f,l)) & (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

[7] The capacity is not defined on an algebra of $\Delta(S)$ as usual, since first the sets not in $\mathcal{D}$ is irrelavant for our purpose, and second the capacity can alwasys be extended to a whole algebra, but some of its properites like continuity may lose.

represents $\succsim'$. *Moreover, for each* $f \in \mathcal{F}_0$, *the robustness index* $v_f : \mathbb{R} \to [0,1]$ *defined by* $v_f(t) = V(f,l)$ *if* $E_p u = t$ *is continuous when* $v_f(\mathbb{R}) \neq \{0,1\}$.

Given $f \in \mathcal{F}_0$, if there exists $l \in \mathcal{L}_1$ such that $(f,l) \notin \max \mathcal{M} \cup \min \mathcal{M}$, then we say that f is an *ambiguous act*, and l is *ambiguous for* f. Otherwise, we call f an *unambiguous act*. Denote by $\mathcal{F}_a$ the set of all ambiguous acts.

Given $A \subseteq S$, if for all $x, y \in X$, xAy is an unambiguous act, then A is called an *unambiguous event*. Let $\mathcal{U}$ be the set of all unambiguous events. Note that $\emptyset, S \in \mathcal{U}$.

Given a nonempty closed subset C of $\Delta(S)$, we say that a capacity c on $\mathcal{D}$ is *compatible with the information set* C if (1) $c(D_1) \geq c(D_2)$ when $D_1,\ D_2 \in \mathcal{D}$ and $D_1 \cap C \supseteq D_2 \cap C$; and (2) for any $f \in \mathcal{F}_0$, $c(D(f,l)) > 0$ when $\max\{(f,p) \mid p \in C\} \succ l$, and $c(D(f,l)) < 1$ when $l \succ \min\{(f,p) \mid p \in C\}$. The underlying information structure captured by C is that the decision maker believes the true probability lies in C, but he cannot exclude any element of C. In particular, if $C = \{p\}$ for some $p \in \Delta(S)$ and $D \in \mathcal{D}$, then $c(D) = 1$ when $p \in D$ and $c(D) = 0$ otherwise. In this case, the decision maker has a subjective probability p. Every act and every event is unambiguous.

Given $f_1, f_2 \in \mathcal{F}_0$ and $\lambda \in [0,1]$, define their *convex combination* $f_1\lambda f_2$ to be an act in $\mathcal{F}_0$ such that $l_{f_1\lambda f_2(s)} \sim l_{f_1(s)}\lambda l_{f_2(s)}$ for all $s \in S$. Rigorously speaking, $f_1\lambda f_2$ denotes a family of such acts, but we will not need to distinguish among them for our purpose. Given $(f_1,l_1),(f_2,l_2) \in \mathcal{F}_0 \times \mathcal{L}_1$, if $l_{f_1\frac{1}{2}f_2(s)} \sim l_1\frac{1}{2}l_2$ for all $s \in S$, we say that (f_2,l_2) is a *complement pair* of (f_1,l_1), and denote it by $(-f_1,-l_1)$. Again a complement pair indeed denotes a family of such pairs but they will be treated as an equivalent class and serve our purpose in the same way. Note that since X is connected and unboundedness is assumed, the convex combination and the complement pair always exist. Given a utility function u as in Lemma 1, then $u(f_1\lambda f_2(s)) = \lambda u(f_1(s)) + (1-\lambda)u(f_2(s))$ and $u(f_1(s)) + u(-f_1(s)) = E_{l_1}u + E_{-l_1}u$ for all $s \in S$.

A.5′ Belief Consistency. If $(f,l) \in \max \mathcal{M}$ and $l' \succ -l$, then $(-f,l') \in \min \mathcal{M}$. If $(f,l) \in \min \mathcal{M}$ and $-l \succ l'$, then $(-f,l') \in \max \mathcal{M}$. If $(f_1,l_1),(f_2,l_2) \in \min \mathcal{M}$ and $\lambda \in (0,1)$, then $(f_1\lambda f_2, l_1\lambda l_2) \in \min \mathcal{M}$.

A.3.2′ Strong Monotonicity. Let $x_1, x_2 \in X$, $f_1, f_2 \in \mathcal{F}_0$ and $l_1, l_2 \in \mathcal{L}_1$ be given. Then

(1) $(l_1,x_1) \succsim (l_2,x_2)$ if and only if $L_1(x_1) \geq L_2(x_2)$;

(2) if $(f_2,p) \succsim l_2$ implies $(f_1,p) \succsim l_1$ either for each $p \in \Delta(S)$ or for each $p \in D(f_3,l_3)$ where $(f_3,l_3) \in \max \mathcal{M}$, then $(f_1,l_1) \succsim'(f_2,l_2)$.

(3) if $f \in \mathcal{F}_a$ and l_1, l_2 are ambiguous for f, then $l_2 \succ l_1$ implies $(f, l_1) \succ' (f, l_2)$.

Lemma 3. *Suppose that $\succsim$ satisfies Axiom 1, 2.1 - 4.1, 5, and u is given as in Lemma 1. The following statements are equivalent.*

(1) $\succsim'$ satisfies Axiom 1', 2', 3.2', 4', 5'.

(2) There exists a unique continuous capacity c on $\mathcal{D}$ such that

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l, x) \in \mathcal{L}_1 \times X \\ c(D(f, l)) & (f, l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

represents $\succsim'$. For each $f \in \mathcal{F}_a$, v_f defined in Lemma 2 is continuous and strictly decreasing on $v_f^{-1}((0,1))$. Besides, there exists a unique nonempty closed convex set $C \subseteq \Delta(S)$ such that c is compatible with the information set C. Moreover $\mathcal{U}$ is a λ-system and $\mathcal{U} = \{A \subseteq S \mid p(A) = p'(A)$ for all $p, p' \in C\}$.

Suppose the hypothesis of Lemma 3 and statement (1). Then for any $f \in \mathcal{F}_a$, $\min_{p\in C} u(f, p) = \max_{(f,l)\in \max \mathcal{M}} E_p u$ and $\max_{p\in C} u(f, p) = \min_{(f,l)\in \min \mathcal{M}} E_p u$. Moreover,

$$v_f(t) \begin{cases} = 1 & t \leq \min_{p\in C} u(f, p) \\ \in (0, 1) & \min_{p\in C} u(f, p) < t < \max_{p\in C} u(f, p) \\ = 0 & t \geq \max_{p\in C} u(f, p) \end{cases}$$

and v_f is strictly decreasing on $(\min_{p\in C} u(f, p), \max_{p\in C} u(f, p))$. For any $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, $\min_{p\in C} u(f, p) = \max_{p\in C} u(f, p) = \max_{(f,l)\in \max \mathcal{M}} E_p u = \inf_{(f,l)\in \min \mathcal{M}} E_p u$, and

$$v_f(t) = \begin{cases} 1 & t \leq \min_{p\in C} u(f, p) \\ 0 & t > \min_{p\in C} u(f, p). \end{cases}$$

Given $f \in \mathcal{F}_0$, we denote by $\underline{l}_f \in \mathcal{L}_1$ the *essential minimum* of f if $E_{\underline{l}_f} u = \min_{p\in C} u(f, p)$, and $\bar{l}_f \in \mathcal{L}_1$ the *essential maximum* of f if $E_{\bar{l}_f} u = \max_{p\in C} u(f, p)$. We call $[E_{\underline{l}_f} u, E_{\bar{l}_f} u]$ the *essential range* of f.

5 Preferences on $\mathcal{F}_0$

A.2.2. Neutrality. If $(f, p) \in \mathcal{F}_1$ generates l_x for some $x \in X$, then $(f, p) \sim f_x$.

A.3.2. Separability. For any $f \succ g$ in $\mathcal{F}_0$, there exists $l \in \mathcal{L}_1$ such that $f \succ l \succ g$.

A.4.2. Bound independence. There exist $\alpha, \beta, \gamma \in (0,1)$ such that for any $f \in \mathcal{F}_a$, $g \in \mathcal{F}_0 \setminus \mathcal{F}_a$, and $x \prec y$ in X,

$$\beta g + (1-\beta) f_x \succsim \alpha f + (1-\alpha) f_x \implies \beta g + (1-\beta) f_y \succsim \alpha f + (1-\alpha) f_y, \tag{6}$$

$$\alpha f + (1-\alpha) f_x \succsim \gamma g + (1-\gamma) f_x \implies \alpha f + (1-\alpha) f_y \succsim \gamma g + (1-\gamma) f_y. \tag{6'}$$

Axiom A.4.2 is much weaker than the certainty independence axioms in the literature. Below are two types of them presented in our context.

A.2. Certainty independence. (Gilboa and Schmeidler (GS), 1989) For any $f, g \in \mathcal{F}_0$, $x \in X$ and $\alpha \in (0,1)$,

$$f \succsim g \iff \alpha f + (1-\alpha) f_x \succsim \alpha g + (1-\alpha) f_x. \tag{2}$$

By Maccheroni, Marinacci, and Rustichini (MMR, 2006), it is equivalent to the following. For any $f, g \in \mathcal{F}_0$, $x, y \in X$ and $\alpha, \beta \in (0,1]$,

$$\alpha f + (1-\alpha) f_x \succsim \alpha g + (1-\alpha) f_x \implies \beta f + (1-\beta) f_y \succsim \beta g + (1-\beta) f_y. \tag{3}$$

MMR (2006) further argues that this axiom actually involves two types of independence: independence relative to mixing with constant acts and independence relative to the weights used in such mixing. MMR (2006) weakens the second type of independence and propose the following axiom.

A.2. Weak certainty independence. (MMR, 2006) For any $f, g \in \mathcal{F}_0$, $x, y \in X$ and $\alpha \in (0,1)$,

$$\alpha f + (1-\alpha) f_x \succsim \alpha g + (1-\alpha) f_x \implies \alpha f + (1-\alpha) f_y \succsim \alpha g + (1-\alpha) f_y. \tag{4}$$

In both papers, along with other axioms, an act f is identified with a state-dependent utility vector $u(f)$ and is evaluated by a functional W on all such utility vectors, i.e., $W(f) = I(u(f))$. Given a utility vector, $\lambda \in \mathbb{R}_+$ and $t \in \mathbb{R}$, we write $\lambda u + t$ for simplicity to denote $\lambda u + te$ where e is the unit vector. The influence of Axiom 2 in GS (1989) is that $I(\lambda u + t) = \lambda I(u) + t$, while that in MMR (2006) implies that $I(\lambda u + t) = I(\lambda u) + t$. MMR (2006) provides a thought experiment to show their different behavioral implication.

Table 1:

$t > 0$	Black	White
f_t	t	t
g_t	3t	0.01t

Example 2 (MMR, 2006): Consider an urn that contains 90 black and white balls in unknown proportion, and the following bets (payoffs are in dollars). That is, f_t pays t dollars whatever happens, while g_t pays $3t$ dollars if a black ball is drawn and t cents otherwise. For example, $t = 10$ and $t = 10^4$.

Table 2:

10	Black	White
f_{10}	10	10
g_{10}	30	0.1

Table 3:

10^4	Black	White
f_{10^4}	10,000	10,000
g_{10^4}	30,000	100

Under reasonable assumption of the decision maker's Bernoulli utility function, GS (1989) implies that either $f_t \succsim g_t$ for all t or $g_t \succsim f_t$ for all t, while MMR (2006) suggests that there may exist $\bar{t}$ such that $f_t \succsim g_t$ when $t \geq \bar{t}$ and $g_t \succsim f_t$ when $t \leq \bar{t}$.

MMR's axiom weakens GS's and breaks down the positive homogeneity of W, but it keeps the translation invariance. Axiom 4.2 further weakens Axiom 2 of MMR and breaks down the translation invariance. Consider the following analogous thought experiment.

The act f_t pays $50 + t$ dollars whatever happens, while g_t pays $100 + t$ dollars when a black ball is drawn and t dollars otherwise. For example, $t = 0$ and $t = 10^4$.

Table 4:

$t \geq 0$	Black	White
f_t	50+t	50+t
g_t	100+t	t

Table 5:

0	Black	White
f_0	50	50
g_0	100	0

Table 6:

10^4	Black	White
f_{10^4}	10,050	10,050
g_{10^4}	10,100	10,000

Both GS (1989) and MMR (2006) imply that either $f_t \succsim g_t$ for all t or $g_t \succsim f_t$ for all t, while Axiom 4.2 suggests that there may exist $\bar{t}$ such that $f_t \succsim g_t$ when $t \leq \bar{t}$ and $g_t \succsim f_t$ when $t \geq \bar{t}$.

In fact, along with the other axioms, Axiom 4.2 implies that there exist $k, k' > 0$ such that $I(u(f) + t) - I(u(f)) \in [k't, kt]$ for $f \in \mathcal{F}_a$ and $t > 0$, while $I(u(f) + t) = I(u(f)) + t$ for $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$ and $t \in \mathbb{R}$. If $\beta = \gamma$ in Axiom 4.2, then $k = k'$ and $I(u(f) + t) = I(u(f)) + kt$ for $f \in \mathcal{F}_a$ and $t \in \mathbb{R}$. If $\alpha = \beta = \gamma$, then $k = k' = 1$ and $I(u(f) + t) = I(u(f)) + t$ for all $f \in \mathcal{F}_0$ and $t \in \mathbb{R}$, which is the case in MMR (2006).

Axiom 4.2 relaxes the independence axioms in two folds. First, it differentiates the effects of the certainty part on ambiguous acts and unambiguous acts. Both (6) and (6′) allow that a change in the certainty part of an ambiguous act is different in magnitude from a change in that of an unambiguous act. This is the consequence of different combination coefficients of the same constant part but with different types of acts. For example, in (6), an improvement from f_x to f_y by $1-\alpha$ proportion in an ambiguous act may equal to that by $1-\beta$ proportion in an unambiguous act. Hence, Axiom 4.2 the decision maker perceives the certainty part differently in different types of acts.

Second, it allows a range of possible effects of the certainty part on ambiguous acts rather than a particular one. While (6) implies that an improvement from f_x to f_y by $1-\alpha$ proportion in an ambiguous act will not exceed that by $1-\beta$ proportion in an unambiguous act, (6′) implies that the improvement will not be weaker than that by $1-\gamma$ proportion in an unambiguous act. While the other axioms normalize the effects of changes in the certainty part in unambiguous acts, Axiom 4.2 uses that to measure the effects on ambiguous acts and gives a uniform bound.

Let $w : \mathbb{R} \times [0,1] \to [-\infty, \infty)$ be given. We say w is normalized if $w(u, 1) = u$ for all $u \in \mathbb{R}$. Let $u : X \to \mathbb{R}$ and $W : \mathcal{F}_0 \to \mathbb{R}$ be given. We say that W is *bounded in translation* if there exists $k, k' > 0$ such that for all $f, g \in \mathcal{F}_a$ and $t > 0$, $u(f) = u(g) + t$ implies $W(f) - W(g) \in [k't, kt]$.

A.6. Dominance. For any $f_1, f_2 \in \mathcal{F}_0$, if $(f_1, l) \succsim' (f_2, l)$ for all $l \in \mathcal{L}_1$, then $f_1 \succsim f_2$. For any $N \in \mathbb{Z}_+$, $f, f_1, ..., f_N \in \mathcal{F}_a$, if $\max\{\underline{l}_{f_n} \mid n = 1, ..., N\} \succ \underline{l}_f$ and $\max\{(f_n, l) \mid i = 1, ..., N\} \succ' (f, l)$ for all l such that $\bar{l}_f \succsim l \succ \underline{l}_f$, then $\max\{f_n \mid i = 1, ..., N\} \succ f$.

Theorem 1. *The following statements are equivalent.*

(1) $\succsim$ *satisfies Axiom 1 - 6, and* $\succsim'$ *satisfies Axiom 1', 2', 3.2', 4', 5'.*

(2) (I) There exists a continuous function $u : X \to \mathbb{R}$ *unique up to a positive affine transformation, a unique continuous capacity* c *on* $\mathcal{D}$ *and a greatest normalized increasing and upper semicontinuous function* $w : \mathbb{R} \times [0,1] \to [-\infty, \infty)$ *such that*

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l,x) \in \mathcal{L}_1 \times X \\ c(D(f,l)) & (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

represents $\succsim'$ *and*

$$W(\cdot) = \begin{cases} E_l u & l \in \mathcal{L}_1 \\ \max_{t \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]} w(t, v_f(t)) & f \in \mathcal{F}_0 \end{cases}$$

represents $\succsim$*, where* v_f *is continuous and strictly decreasing on* $[E_{\underline{l}_f} u, E_{\bar{l}_f} u]$ *when* $f \in \mathcal{F}_a$*, while* W *is bounded in translation.*

(II) There exists a unique non empty closed convex set $C \subseteq \Delta(S)$ such that c is compatible with the information set C. Moreover, $\mathcal{U}$ is a λ-system and $\mathcal{U} = \{A \subseteq S \mid p(A) = p'(A)$ for all $p,\ p' \in C\}$.

One can show that if $w(u,t) = -\infty$ for some $(u,t) \in \mathbb{R} \times [0,1]$, then $w(u',t) = -\infty$ for all $u' \in \mathbb{R}$. This means that the decision maker has a threshold value of robustness, below which he regards it as unacceptable. The maximin expected utility (MEU) in GS (1989) is an example. A MEU decision maker evaluates an act by its minimum expected utility among a nonempty convex closed set of priors. This set can be viewed as his information set in our framework, and his robustness preference is

$$w(u,t) = \begin{cases} u & t = 1 \\ -\infty & t < 1. \end{cases}$$

Hence, a MEU decision maker only consider the possible aspiration level that has the full robustness, i.e., the minimum expected utility level.

6 Appendix

Before proving Lemma 1, we provide an analogy of the classic expected utility representation result with the preference $\succsim$ defined on $\mathcal{L}_1 = \{l \in \mathcal{L}_0 \mid |\text{supp}(l)| \leq |S|\}$. The following axioms are standard except that here they apply to $\succsim$ when the relevant alternatives exist in $\mathcal{L}_1$.

B.1. Weak Order. $\succsim$ is complete and transitive.

B.2. Independence. For any $l_1, l_2, l_3 \in \mathcal{L}_1$ and $\lambda \in (0,1)$, $l_1 \succ l_2$ implies that $l_1 \lambda l_3 \succ l_2 \lambda l_3$ if they exist in $\mathcal{L}_1$.

B.3. Continuity For any $l_1 \succ l_2 \succ l_3$ in $\mathcal{L}_1$, if $l_1 \lambda l_2$ exists for some $\lambda \in (0,1)$, then $l_1 \alpha l_3 \succ l_2$ and $l_2 \succ l_1 \beta l_3$ for some $\alpha, \beta \in (0,1)$.

Lemma 4. *A preference $\succsim$ satisfies B.1, B.2 and B.3 if and only if there is a function $u : X \to \mathbb{R}$ such that $l \succsim l' \Leftrightarrow E_l u \geq E_{l'} u$ for all $l, l' \in \mathcal{L}_1$. Moreover, u is unique up to a positive affine transformation.*

Proof. The necessity is obvious. We show the sufficiency in three steps. The proof is based on that of Theorem 8.3 and 8.4 in Fishburn (1970). The only difference here is to check which properties hold without the "mixture set" assumption and whether they are sufficient to derive an expected utility representation.

Step 1. If $\succsim$ satisfies B.1, B.2 and B.3, then the following holds for all $l_1, l_2, l_3 \in \mathcal{L}_1$.

- **C.1.** Suppose that $l_1 \succ l_2$ and $0 \leq \alpha < \beta \leq 1$. Then $l_1\beta l_2 \succ l_1\alpha l_2$ if they exist in $\mathcal{L}_1$.
- **C.2.** Suppose that $l_1 \succsim l_2$, $l_2 \succsim l_3$ and $l_1 \succ l_3$. If $l_1\lambda l_3$ exists for some $\lambda \in (0, 1)$, then $l_2 \sim l_1\alpha l_3$ for exactly one $\alpha \in [0, 1]$.
- **C.3.** Suppose that $l_1 \sim l_2$ and $0 \leq \alpha \leq 1$. Then $l_1\alpha l_2 \sim l_1$ if $l_1\alpha l_2$ exists in $\mathcal{L}_1$.
- **C.4.** Suppose that $l_1 \sim l_2$ and $0 \leq \alpha \leq 1$. Then $l_1\alpha l_3 \sim l_2\alpha l_3$ if they exist in $\mathcal{L}_1$.

The proof of C.1 and C.2 is exactly the same as Fishburn's proof. To check C.3, the case when $\alpha = 0$ or 1 is easy. Suppose that $0 < \alpha < 1$, $l_1\alpha l_2$ exists in $\mathcal{L}_1$ and $l_1\alpha l_2 \succ l_1$. Then by B.2, we have $(l_1\alpha l_2)\alpha l_2 \succ l_1\alpha l_2$. Since $l_1\alpha l_2 \succ l_2$, by C.1, $l_1\alpha l_2 \succ (l_1\alpha l_2)\alpha l_2$ which is a contradiction. The case when $l_1 \succ l_1\alpha l_2$ can lead to a similar contradiction. Hence, $l_1\alpha l_2 \sim l_1$.

For C.4, it holds obviously when $\alpha = 0$ or 1, or $l_3 \sim l_1$. Suppose that $0 < \alpha < 1$, $l_1\alpha l_3$ and $l_2\alpha l_3$ exist in $\mathcal{L}_1$, and $l_3 \succ l_1$. Then by C.1, $l_3 \succ l_1\alpha l_3 \succ l_1$. Thus $l_3 \succ l_1\alpha l_3 \succ l_2$, and by C.1, $l_1\alpha l_3 \sim l_2\beta l_3$ for some $\beta \in [0, 1]$. Suppose that $\beta < \alpha$. By C.1, $l_2\frac{\beta}{\alpha}l_3 \succ l_2 \sim l_1$. By B.2, $(l_2\frac{\beta}{\alpha}l_3)\alpha l_3 \succ l_1\alpha l_3$. Then $l_2\beta l_3 = (l_2\frac{\beta}{\alpha}l_3)\alpha l_3 \succ l_1\alpha l_3$ which is a contradiction. Similarly, it cannot be that $\beta > \alpha$. Thus, $l_1\alpha l_3 \sim l_2\alpha l_3$.

Step 2. Assume that $l_x \succ l_y$ for some $x, y \in X$. Let $l_x l_y = \{l \in \mathcal{L}_1 \mid l_x \succsim l \succsim l_y\}$. Then there exists a function $f : l_x l_y \to [0, 1]$ such that for all $l, l' \in l_x l_y$, (1) $l \succsim l'$ if and only if $f(l) \geq f(l')$, and (2) for all $\alpha \in [0, 1]$, $f(l\alpha l') = \alpha f(l) + (1 - \alpha) f(l')$ if $l\alpha l'$ exists in $\mathcal{L}_1$.

For all $l \in l_x l_y$, let $f(l)$ to be the unique number in $[0, 1]$ such that $l \sim l_x f(l) l_y$. The function f is well-defined by C.2. By C.1, $l_x f(l) l_y \succsim l_x f(l') l_y$ if and only if $f(l) \geq f(l')$. Thus, (1) holds by the definition of f.

To check (2), the case when $\alpha = 0$ or 1 is obvious. Suppose that $0 < \alpha < 1$ and $l\alpha l'$ exists in $\mathcal{L}_1$. Let $z, w \in \text{supp}(l) \cup \text{supp}(l')$ be given such that $l_z \succsim l_r \succsim l_w$ for all $r \in \text{supp}(l) \cup \text{supp}(l')$.

By repeatedly using C.1 or C.3, we have $l_z \succsim l \succsim l_w$. Thus by C.2 or C.3, $l \sim l_z\beta l_w$ for some $\beta \in [0,1]$. Similarly, $l' \sim l_z\gamma l_w$ for some $\gamma \in [0,1]$. Note that $(l_z\beta l_w)\alpha l'$ also exists in $\mathcal{L}_1$. By C.4, $l\alpha l' \sim (l_z\beta l_w)\alpha l'$. Similarly, $(l_z\beta l_w)\alpha l' \sim (l_z\beta l_w)\alpha(l_z\gamma l_w)$. Hence, $l\alpha l' \sim (l_z\beta l_w)\alpha(l_z\gamma l_w)$.

If $l \sim l'$, then (2) holds trivially. Suppose without loss of generality that $l \succ l'$. Then $l_z \succ l_w$, $\beta > \gamma$ and $f(l) > f(l')$. Note that $l_x \succsim l \sim l_z\beta l_w \succ l_w$. By C.2, $l_z\beta l_w \sim l_x\beta' l_w$ for a unique $\beta' \in [0,1]$. Similarly, $l_z\gamma l_w \sim l_x\gamma' l_w$ for a unique $\gamma' \in [0,1]$, and $\beta' > \gamma'$. Note that $l_z\gamma l_w = (l_z\beta l_w)\frac{\gamma}{\beta}l_w \sim (l_x\beta' l_w)\frac{\gamma}{\beta}l_w = l_x\frac{\beta'\gamma}{\beta}l_w$ by C.4. Then the uniqueness of γ' implies that $\gamma' = \frac{\beta'\gamma}{\beta}$. Hence, $(l_z\beta l_w)\alpha(l_z\gamma l_w) = (l_z\beta l_w)\alpha[(l_z\beta l_w)\frac{\gamma}{\beta}l_w] = (l_z\beta l_w)[\alpha + (1-\alpha)\frac{\gamma}{\beta}]l_w \sim (l_x\beta' l_w)[\alpha + (1-\alpha)\frac{\gamma}{\beta}]l_w = (l_x\beta' l_w)[\alpha + (1-\alpha)\frac{\gamma'}{\beta'}]l_w = (l_x\beta' l_w)\alpha(l_x\gamma' l_w)$.

Analogously, $l_x\beta' l_w \sim l_x f(l) l_y$ and $l_x\gamma' l_w \sim l_x f(l') l_y$. Since $l_x\beta' l_w = l_x\frac{\beta'-\gamma'}{1-\gamma'}(l_x\gamma' l_w) \sim l_x\frac{\beta'-\gamma'}{1-\gamma'}[l_x f(l') l_y]$, $l_x f(l) l_y = l_x\frac{f(l)-f(l')}{1-f(l')}[l_x f(l') l_y]$ and $l_x \succ l_x f(l') l_y$, then $\frac{\beta'-\gamma'}{1-\gamma'} = \frac{f(l)-f(l')}{1-f(l')}$ by C.1. Hence, $(l_x\beta' l_w)\alpha(l_x\gamma' l_w) = l_x\frac{\alpha(\beta'-\gamma')}{1-\gamma'}(l_x\gamma' l_w) \sim l_x\frac{\alpha(\beta'-\gamma')}{1-\gamma'}[l_x f(l') l_y] = l_x\frac{\alpha[f(l)-f(l')]}{1-f(l')}[l_x f(l') l_y] = [l_x f(l) l_y]\alpha[l_x f(l') l_y]$. Therefore, we get that $l\alpha l' \sim [l_x f(l) l_y]\alpha[l_x f(l') l_y] = l_x[\alpha f(l) + (1-\alpha) f(l')]l_y$, and thus by the definition of f, $f(l\alpha l') = \alpha f(l) + (1-\alpha) f(l')$.

Step 3. There exsits a function $u : X \to \mathbb{R}$ such that $l \succsim l' \Leftrightarrow E_l u \geq E_{l'} u$ for all $l, l' \in \mathcal{L}_1$.

If there are no $x, y \in X$ such that $l_x \succ l_y$, $l \sim l'$ for all $l, l' \in \mathcal{L}_1$ by repeatedly using C.3. Thus any constant u works. Suppose that there exists $l_x \succ l_y$ for some $x, y \in X$. For $i = 1$ or 2, let $l_{x_i}l_{y_i} = \{l \in \mathcal{L}_1 \mid l_{x_i} \succsim l \succsim l_{y_i}\}$ such that $l_x l_y \subseteq l_{x_i}l_{y_i}$. For both i, let $f_i' : l_{x_i}l_{y_i} \to [0,1]$ be the function constructed as in Step 2, let f_i be the affine transformation of f_i' such that $f_i(l_x) = 1$ and $f_i(l_y) = 0$, and thus f_i still satisfies (1) and (2) in Step 2.

Let $l \in l_{x_1}l_{y_1} \cap l_{x_2}l_{y_2}$. If $l \sim l_x$ or $l \sim l_y$, then $f_1(l) = f_2(l)$. Otherwise, one of the following cases must be true: $l_{x_i} \succsim l \succ l_x \succ l_y$, $l_x \succ l \succ l_y$ or $l_x \succ l_y \succ l \succsim l_{y_i}$. Consider the first case. Suppose without loss of generality that $l_{x_1} \succsim l_{x_2}$. Then for both i, $l_{x_2} \in l_{x_i}l_{y_i}$, and $l_x \sim l_{x_2}\alpha l_y$ for a unique $\alpha \in (0,1)$ by C.2. Hence, $1 = f_i(l_x) = \alpha f_i(l_{x_2}) + (1-\alpha) f_i(l_y) = \alpha f_i(l_{x_2})$ for $i = 1, 2$. Since $l \sim l_{x_2}\beta l_y$ for a unique $\beta \in (0,1]$, then $f_i(l) = \beta f_i(l_{x_2}) = \frac{\beta}{\alpha}$ for $i = 1, 2$. Similarly, in the other cases, we also get $f_1(l) = f_2(l)$.

For all $l \in \mathcal{L}_1$, let $f(l)$ be the common value of $f_i(l)$ defined on every such $l_{x_i}l_{y_i}$ as above. Since each pair of $l, l' \in \mathcal{L}_1$ is contrained in some $l_{x_i}l_{y_i}$, then f satisfies condition (1) and (2) in Step 2. Define $u : X \to \mathbb{R}$ by $u(x) = f(l_x)$ for all $x \in X$. Finally, for all $l \in \mathcal{L}_1$,

$$f(l) = f[\sum_{x\in\text{supp}(l)} l(x)l_x] = \sum_{x\in\text{supp}(l)} l(x)f(l_x) = \sum_{x\in\text{supp}(l)} l(x)u(x) = E_l u.$$

We complete the proof by checking the uniqueness property. Let $v : X \to \mathbb{R}$ be such that $l \succsim l' \Leftrightarrow E_l v \geq E_{l'} v$ for all $l, l' \in \mathcal{L}_1$. If there is no $l_x \succ l_y$ in X, then u and v constant on X. Clearly, u is an affine transformation of v. Otherwise, fix some $x, y \in X$ such that $l_x \succ l_y$. Let $u'(z) = \frac{u(z)-u(y)}{u(x)-u(y)}$ and $v'(z) = \frac{v(z)-v(y)}{v(x)-v(y)}$ for all $z \in X$. Note that u' and v' are affine transformations of u and v, so for all $l \in \mathcal{L}_1$, $l \succsim l' \Leftrightarrow E_l u' \geq E_{l'} u' \Leftrightarrow E_l v' \geq E_{l'} v'$. Besides, $u'(x) = v'(x) = 1$ and $u'(y) = v'(y) = 0$. Fix $z \in X$. Then one of the following cases is true: $l_z \succsim l_x \succ l_y$, $l_x \succ l_z \succ l_y$ or $l_x \succ l_y \succsim l_z$. Using the similar argument as before, we get $u'(z) = v'(z)$ in all cases. Thus $v(z) = \frac{v(x)-v(y)}{u(x)-u(y)}u(z) + v(y) - \frac{v(x)-v(y)}{u(x)-u(y)}u(y)$ for all $z \in X$. This shows that v is an affine transformation of u. □

Using Lemma 4, we can prove Lemma 1. Let $\mathbb{Z}_+$ denote the set of positive integers.

Proof of Lemma 1. The necessity part is easy. Let us check the sufficiency. By A.2.1, we only need to show the analogous representation result for $\succsim$ on $\mathcal{L}_1$.

Clearly A.1 implies B.1. We show that A.3.1 implies B.3. Let $l_1 \succ l_2 \succ l_3$ in $\mathcal{L}_1$ be given and suppose that $l_1 \lambda l_3$ exists for some $\lambda \in (0, 1)$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of elements in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. For any continuous real-valued function η on X, $\lim_{n\to\infty} \int \eta d(l_1 \lambda_n l_3) = \int \eta dl_1$. Hence, $\lim_{n\to\infty} l_1 \lambda_n l_3 = l_1$. Since $l_1 \succ l_2$ and $\{l \in \mathcal{L}_1 \mid l \succ l_2\}$ is open by A.3.1, then there exists $N \in \mathbb{Z}_+$ such that $l_1 \lambda_N l_3 \succ l_2$. Similarly, pick a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of elements in $(0, 1)$ converging to 0, then there is $N' \in \mathbb{Z}_+$ such that $l_2 \succ l_1 \lambda_N l_3$.

By applying Lemma 4, there exists a function $u : X \to \mathbb{R}$ such that $l \succsim l' \Leftrightarrow E_l u \geq E_{l'} u$ for all $l, l' \in \mathcal{L}_1$. Moreover, u is unique up to a positive affine transformation. To show that u is continuous, suppose the contrary that there exist $\epsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in X such that $\lim_{n\to\infty} x_n = x_0$, $x_0 \in X$, and $|u(x_n) - u(x_0)| > \epsilon$ for all n. Then there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that either $u(x_{n_j}) > u(x_0) + \epsilon$ for all j or $u(x_{n_j}) < u(x_0) - \epsilon$ for all j. Assume the former case and let $\underline{u} = \inf\{u(x_{n_j}) \mid j \in \mathbb{Z}_+\}$. Pick $J \in \mathbb{Z}_+$ such that $u(x_{n_J}) < \underline{u} + \epsilon$. Let $l = l_{x_0} \frac{1}{2} l_{x_{n_J}}$. For all $j \in \mathbb{Z}_+$, $u(x_{n_j}) \geq \underline{u}$ and $\frac{1}{2}u(x_0) + \frac{1}{2}u(x_{n_J}) < \frac{1}{2}(\underline{u} - \epsilon) + \frac{1}{2}(\underline{u} + \epsilon) = \underline{u}$, so $l_{x_{n_j}} \succsim l$. For any continuous real-valued function η on X, $\lim_{j\to\infty} \int \eta dl_{x_{n_j}} = \lim_{j\to\infty} \eta(x_{n_j}) = \eta(x_0) = \int \eta dl_{x_0}$. Thus $\lim_{j\to\infty} l_{x_{n_j}} = l_{x_0}$. By A.3, $l_{x_0} \succsim l$, which contradicts that $u(x_0) < \frac{1}{2}u(x_0) + \frac{1}{2}\underline{u} \leq \frac{1}{2}u(x_0) + \frac{1}{2}u(x_{n_J})$. The argument follows analogously for the case when $u(x_{n_j}) < u(x_0) - \epsilon$ for all j.

Lastly, if $u(X) = \mathbb{R}$, then A.5 obviously holds. For the other direction, assume A.5 holds. Then we have $l_x \succ l_y$ such that for all $\lambda \in (0,1)$, there exists $z_1 \in X$ such that $u(y) > \lambda u(z_1) + (1-\lambda)u(x)$, i.e., $u(z_1) < \frac{1}{\lambda}[u(y) - (1-\lambda)u(x)]$. Thus, $\lim_{\lambda \to 0} u(z_1) = -\infty$. Similarly, $u(X)$ is not bounded above. Since u is continuous, X is connected and $u(X)$ is unbounded, then $u(X) = \mathbb{R}$.

□

Proof of Lemma 2. Suppose that (1) holds. Fix $x_0, x_1 \in X$ such that $l_{x_1} \succ l_{x_0}$.

Step 1. If $l_y \succsim l_x$ ($l_x \succ l_y$) for all $y \in \text{supp}(l)$, then $(l,x) \in \max \mathcal{M}$ ($(l,x) \in \min \mathcal{M}$). If $(f,p) \succsim l$ ($l \succ (f,p)$) for all $p \in \Delta(S)$, then $(f,l) \in \max \mathcal{M}$ ($(f,l) \in \min \mathcal{M}$).

By A.3.1′, if $l_y \succsim l_x$ ($l_x \succ l_y$) for all $y \in \text{supp}(l)$, then $(l,x) \in \max \mathcal{L}_1 \times X$ ($(l,x) \in \min \mathcal{L}_1 \times X$), and if $(f,p) \succsim l$ ($l \succ (f,p)$) for all $p \in \Delta(S)$, then $(f,l) \in \max \mathcal{F}_0 \times \mathcal{L}_1$ ($(f,l) \in \min \mathcal{F}_0 \times \mathcal{L}_1$). In particular, $(f_{x_0}, l_{x_0}) \in \max \mathcal{F}_0 \times \mathcal{L}_1$, $(f_{x_0}, l_{x_1}) \in \min \mathcal{F}_0 \times \mathcal{L}_1$, $(l_{x_0}, x_0) \in \max \mathcal{L}_1 \times X$, and $(l_{x_0}, x_1) \in \min \mathcal{L}_1 \times X$. Combining these facts and A.4’, we get that the desired results.

Step 2. For any $(f,l) \in \mathcal{F}_0 \times \mathcal{L}_1$, there is a unique $\lambda \in [0,1]$ such that $(f,l) \sim' (l_{x_1} \lambda l_{x_0}, x_1)$.

Note that $(l_{x_1} \alpha l_{x_0}, x_1) \succ' (l_{x_1} \beta l_{x_0}, x_1)$ if and only if $\alpha > \beta$ in $[0,1]$. Moreover, $(l_{x_0}, x_1) \in \min \mathcal{M}$ and $(l_{x_1}, x_1) \in \max \mathcal{M}$. Hence, if $(f,l) \in \min \mathcal{M} \cup \max \mathcal{M}$, then the unique λ is either 0 or 1. On the other hand, if $(l_{x_1}, x_1) \succ' (f,l) \succ' (l_{x_0}, x_1)$, then by *A*.2′(1) there exists a unique $\lambda \in (0,1)$ such that for all α, β $(0,1)$ with $1 \geq \alpha > \lambda > \beta \geq 0$, $(l_{x_1} \alpha l_{x_0}, x_1) \succ' (f,l) \succ' (l_{x_1} \beta l_{x_0}, x_1)$. If $(l_{x_1} \lambda l_{x_0}, x_1) \succ (f,l)$, then again by *A*.2′, there exists $\mu \in (0,1)$ such that $(l_{x_1} \lambda\mu l_{x_0}, x_1) \succ' (f,l)$, which is a contradiction since $\lambda\mu < \lambda$. Similarly, it cannot be true that $(f,l) \succ' (l_{x_1} \lambda l_{x_0}, x_1)$. Hence, $(f,l) \sim' (l_{x_1} \lambda l_{x_0}, x_1)$.

Step 3. For all $(f,l) \in \mathcal{F}_0 \times \mathcal{L}_1$, define $c(D(f,l)) = \lambda$ if $(f,l) \sim' (l_{x_1} \lambda l_{x_0}, x_1)$, $\lambda \in [0,1]$. Clearly, $c : \mathcal{D} \to \mathbb{R}$ is well defined and it is the unique function on $\mathcal{D}$ such that

$$V(\cdot,\cdot) = \begin{cases} L(x) & (l,x) \in \mathcal{L}_1 \times X \\ c(D(f,l)) & (f,l) \in \mathcal{F}_0 \times \mathcal{L}_1 \end{cases}$$

represents $\succsim'$ on $\mathcal{M}$. We would like to show that c is a continuous capacity on $\mathcal{D}$.

First, c is a capacity on $\mathcal{D}$. Note that $c(\emptyset) = c(D(f_{x_0}, l_{x_1})) = 0$ since $(f_{x_0}, l_{x_1}) \sim' (l_{x_0}, x_1)$, and $c(\Delta(S)) = c(D(f_{x_1}, l_{x_1})) = 1$ since $(f_{x_1}, l_{x_1}) \sim' (l_{x_1}, x_1)$. If $D(f, l) \supseteq D(f', l')$, then $(f, l) \succsim' (f', l')$. Suppose that $(f, l) \sim' (l_{x_1} \lambda l_{x_0}, x_1)$ and $(f', l') \sim' (l_{x_1} \lambda' l_{x_0}, x_1)$. Hence, $\lambda \geq \lambda'$ and thus $c(D(f, l)) \geq c(D(f', l'))$.

To check c is upper continuous, let $\{D(f_n, l_n)\}_{n=1}^{\infty}$ be a non-increasing sequence of sets in $\mathcal{D}$ and $\cap_{n=1}^{\infty} D(f_n, l_n) = D(f, l) \in \mathcal{D}$. We want to show that $\lim_{n\to\infty} c(D(f_n, l_n)) = c(D(f, l))$. Since c is monotone and bounded, then $\underline{c} := \lim_{n\to\infty} c(D(f_n, l_n)) = \inf\{c(D(f_n, l_n)) \mid n \in \mathbb{Z}_+\} \in [0, 1]$. Note that $\underline{c} \geq c(D(f, l))$. If $\underline{c} = 0$, then $\underline{c} = c(D(f, l)) = 0$. Suppose that $\underline{c} > 0$. Thus, $D(f_n, l_n) \neq \emptyset$ for each n. Pick $p_n \in \arg\min_{p \in D(f_n, l_n)} E_p u(f)$ and let $l'_n = (f, p_n)$ for each n. Clearly, $l'_n \succsim l'_m$ when $n \geq m$, which means that $D(f, l'_n) \subseteq D(f, l'_m)$ when $n \geq m$. Moreover, $D(f_n, l_n) \subseteq D(f, l'_n)$, since $p \in D(f_n, l_n)$ implies that $E_p u(f) \geq E_{p_n} u(f) = E_{l_n} u$ and thus $(f, p) \succsim l_n$. Hence, $D(f, l) \subseteq \cap_{n=1}^{\infty} D(f, l'_n)$.

In fact, $D(f, l) = \cap_{n=1}^{\infty} D(f, l'_n)$. Suppose the contrary that there exists $p' \in \cap_{n=1}^{\infty} D(f, l'_n) \setminus D(f, l)$. Thus $l \succ (f, p')$. Pick $l' \in \mathcal{L}_1$ be such that $l \succ l' \succ (f, p')$. Thus $p' \notin D(f, l')$. We check that $D(f_n, l_n) \subseteq D(f, l')$ for some and thus for all sufficiently large $n \in \mathbb{Z}_+$. Suppose that there exists $p'_n \in D(f_n, l_n) \setminus D(f, l')$ for each n. The sequence $\{p'_n\}_{n=1}^{\infty}$ is bounded, so there is a subsequence $\{p'_{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} p'_{n_j} = p^* \in \Delta(S)$. For each $n \in \mathbb{Z}_+$, $p'_{n_j} \in D(f_{n_j}, l_{n_j}) \subseteq D(f_n, l_n)$ for sufficiently large j, and since $D(f_n, l_n)$ is closed, then $p^* \in D(f_n, l_n)$. Thus $p^* \in \cap_{n=1}^{\infty} D(f_n, l_n)$. Since $p'_n \notin D(f, l')$ for each n and $l \succ l'$, then $E_{p^*} u(f) = \lim_{j\to\infty} E_{p'_{n_j}} u(f) \leq E_{l'} u < E_l u$, and thus $p^* \notin D(f, l)$. This is a contradiction to $p^* \in \cap_{n=1}^{\infty} D(f_n, l_n) = D(f, l)$. Hence, $D(f_n, l_n) \subseteq D(f, l')$ for sufficiently large n. Next, note that if $D(f_n, l_n) \subseteq D(f, l')$, then $E_{l'_n} u = \min_{p \in D(f_n, l_n)} E_p u(f) \geq E_{l'} u$, and thus $l'_n \succsim l'$. Combining the facts above, we have that $D(f, l'_n) \subseteq D(f, l')$ for sufficiently large $n \in \mathbb{Z}_+$. Since $p' \in \cap_{n=1}^{\infty} D(f, l'_n)$, then $p' \in D(f, l')$, which contradicts with $l' \succ (f, p')$.

Let $\underline{c}' := \inf\{c(D(f, l'_n)) \mid n \in \mathbb{Z}_+\} = \lim_{n\to\infty} c(D(f, l'_n))$. Clearly, $\underline{c}' \in [0, 1]$. For each $n \in \mathbb{Z}_+$, by the definition of l'_n, $D(f_n, l_n) \subseteq D(f, l'_n)$, so $\underline{c} \leq \underline{c}'$. Since $c(D(f, l)) \leq \underline{c}$, then it suffices to show that $\underline{c}' = c(D(f, l))$.

Suppose that $D(f, l) = \emptyset$. We want to show that $D(f, l'_n) = \emptyset$ for sufficiently large n and thus $\underline{c}' = c(D(f, l)) = 0$. To show that, assume the opposite that for all $n \in \mathbb{Z}_+$, $D(f, l'_n) \neq \emptyset$ and thus $E_{l'_n} u \leq \max_{s \in S} u(f(s))$. Let $\underline{u} = \sup\{E_{l'_n} u \mid n \in \mathbb{Z}_+\}$. Then $\underline{u} \in \mathbb{R}$ and there exists $\underline{l} \in \mathcal{L}_1$ such that $E_{\underline{l}} u = \underline{u}$. Moreover, $D(f, l) = \cap_{n=1}^{\infty} D(f, l'_n) = D(f, \underline{l})$. For each $n \in \mathbb{Z}_+$, choose

$p_n \in D(f, l'_n)$. Let $\{p_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{p_n\}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} p_{n_j} = \underline{p} \in \Delta(S)$. Hence, $E_{\underline{p}}u(f) = \lim_{j\to\infty} E_{p_{n_j}}u(f) \geq \lim_{j\to\infty} E_{l'_{n_j}}u = \underline{u}$, and thus $\underline{p} \in D(f, \underline{l}) = D(f, l)$. This contradicts with $D(f, l) = \emptyset$.

Next, suppose that $\max_{s\in S} u(f(s)) = \min_{s\in S} u(f(s))$. Then $D(f, l)$ and $D(f, l'_n)$, $n \in \mathbb{Z}_+$, are either or $\Delta(S)$. Hence, if $D(f, l) = \emptyset$, then $D(f, l'_n) = \emptyset$ for sufficiently large n; if $D(f, l) = \Delta(S)$, then $D(f, l'_n) = \Delta(S)$ for all $n \in \mathbb{Z}_+$. In either case, $\underline{c}' = c(D(f, l))$.

Lastly, suppose that $D(f, l) \neq \emptyset$, $\max_{s\in S} u(f(s)) \neq \min_{s\in S} u(f(s))$, and $\underline{c}' > c(D(f, l))$. Then $D(f, l) \neq \Delta(S)$ as well. Choose $\lambda \in (c(D(f, l)), \underline{c}')$. Then $(f, l'_n) \succ' (l_{x_1}\lambda l_{x_0}, x_1) \succ' (f, l)$ for all $n \in \mathbb{Z}_+$. By $A.2'(2)$, for each $n \in \mathbb{Z}_+$, there exists $\beta_n \in (0, 1)$ such that $(l_{x_1}\lambda l_{x_0}, x_1) \succ' (f, l'_n\beta_n l)$. Fix any $N \in \mathbb{Z}_+$. If there exists $m_N \in \mathbb{Z}_+$ such that $l'_{m_N} \succsim l'_N\beta_N l$, then $D(f, l'_{m_N}) \subseteq D(f, l'_N\beta_N l)$, and thus $c(D(f, l'_{m_N})) \leq c(D(f, l_N\beta_N l)) < \lambda < \underline{c}$ which is a contradiction. To see that $l'_{m_N} \succsim l'_N\beta_N l$ for some $m_N \in \mathbb{Z}_+$, suppose the contrary that $l'_N\beta_N l \succ l'_m$ for all $m \in \mathbb{Z}_+$. Hence, $D(f, l'_N\beta_N l) \subseteq D(f, l'_m)$ for all $m \in \mathbb{Z}_+$, and then $D(f, l'_N\beta_N l) \subseteq D(f, l)$. On the other hand, since $(f, l'_N) \succ' (f, l)$, then $l \succ l'_N$ and thus $l \succ l'_N\beta_N l$. Since $D(f, l)$ is neither $\emptyset$ nor $\Delta(S)$, and $\max_{s\in S} u(f(s)) \neq \min_{s\in S} u(f(s))$, then $E_l u \in (\min_{s\in S} u(f(s)), \max_{s\in S} u(f(s))]$. Observe that there must exist $q \in \Delta(S)$ such that $E_{l'_N\beta_N l}u < E_q u(f) < E_l u$. Thus, $q \in D(f, l'_N\beta_N l) \setminus D(f, l)$ which is a contradiction, as desired.

To check that c is lower continuous, let $\{D(f_n, l_n)\}_{n=1}^{\infty}$ be a non-decreasing sequence of sets in $\mathcal{D}$ and $\cup_{n=1}^{\infty} D(f_n, l_n) = D(f, l)$. Note that $D(f, l)$ is the intersection of finitely many half spaces, that is, it is a convex polytope. By the vertex representation of a convex polytope, it can be written as the convex hull of finitely many points of it. Since $\{D(f_n, l_n)\}_{n=1}^{\infty}$ is a non-decreasing sequence of convex sets, then $D(f, l) \subseteq D(f_n, l_n)$ for sufficiently large n. This implies that $D(f, l) = D(f_n, l_n)$ for sufficiently large n. Therefore, $\lim_{n\to\infty} c(D(f_n, l_n)) = c(D(f, l))$.

Step 4. For each $f \in \mathcal{F}_0$, the robustness index $v_f : \mathbb{R} \to [0, 1]$ defined by $v_f(t) = V(f, l)$ if $E_p u = t$ is continuous when $v_f(\mathbb{R}) \neq \{0, 1\}$.

Fix $f \in \mathcal{F}_0$. We first check that v_f is well-defined. For any $t \in \mathbb{R}$, there exists $l \in \mathcal{L}_1$ such that $E_l u = t$, since $u(X) = \mathbb{R}$. Suppose that $l' \in \mathcal{L}_1$, $l' \neq l$ and $E'_l u = t$. Hence, $D(f, l) = D(f, l')$ and thus $V(f, l) = V(f, l')$. Note that v_f is non-increasing, $v_f(t) = 1$ when $t \leq \min_{s\in S} u(f(s))$, and $v_f(t) = 0$ when $t > \max_{s\in S} u(f(s))$.

Next, suppose that $v_f(\mathbb{R}) \neq \{0, 1\}$, and we would like to show that v_f is continuous. In this case, it must be true that $\max_{s\in S} u(f(s)) \neq \min_{s\in S} u(f(s))$. Suppose the opposite that v_f is not continous at $t \in \mathbb{R}$. To begin with, we assume that $v_f(t) \in (0,1)$. Then there exist $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^{\infty}$ of elements in $\mathbb{R}$ such that $\lim_{n\to\infty} t_n = t$ and $|v_f(t) - v_f(t_n)| \geq \epsilon$. Thus there is a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$ such that $t_{n_j} > t$ for all $j \in \mathbb{Z}_+$ or $t_{n_j} < t$ for all $j \in \mathbb{Z}_+$. Suppose the former case. Then $v_f(t_{n_j}) \leq v_f(t) - \epsilon$ for all $j \in \mathbb{Z}_+$. Let $l \in \mathcal{L}_1$ and $l_{n_j} \in \mathcal{L}_1$ for each $j \in \mathbb{Z}_+$ be such that $E_l u = t$ and $E_{l_{n_j}} u = t_{n_j}$. Let $\lambda = v_f(t) - \frac{2}{\epsilon}$. Then $(f,l) \succ' (l_{x_1}\lambda l_{x_2}, x_1) \succ' (f, l_{n_j})$ for all $j \in \mathbb{Z}_+$. By $A.2'(2)$, there exists $\alpha_j \in (0,1)$ for each $j \in \mathbb{Z}_+$ such that $(f, l\alpha_j l_{n_j}) \succ' (l_{x_1}\lambda l_{x_2}, x_1)$. Note also that $l \succ l_{n_j}$ and thus $l\alpha_j l_{n_j} \succ l$, $j \in \mathbb{Z}_+$. Fix $J \in \mathbb{Z}_+$. Since $\lim_{j\to\infty} t_{n_j} = t$, then there exists $K \in \mathbb{Z}_+$ such that $l\alpha_J l_{n_J} \succ l_{n_K} \succ l$. Hence, $(f, l_{n_K}) \succsim' (f, l\alpha_J l_{n_J})$ and thus $(f, l_{n_K}) \succ' (l_{x_1}\lambda l_{x_2}, x_1)$, which contradicts that $(l_{x_1}\lambda l_{x_2}, x_1) \succ' (f, l_{n_j})$ for all $j \in \mathbb{Z}_+$. Suppose the later case where for all $j \in \mathbb{Z}_+$, $t_{n_j} < t$ so that $v_f(t_{n_j}) \geq v_f(t) + \epsilon$. A similar argument as above lead to another contradiction. If $v_f(t) = 1$, then $v_f(t') = 1$ for all $t' \leq t$. Thus $t_n > t$ for all $n \in \mathbb{Z}_+$. Since $v_f(\mathbb{R}) \neq \{0,1\}$, then there exists $N \in \mathbb{Z}_+$ such that $v_f(t_N) > 0$, and thus $(f, l_N) \notin \min \mathcal{M}$. Then $A.2'(2)$ can be used similarly to derive a contradiction. If $v_f(t) = 0$, then $v_f(t') = 0$ for all $t' \geq t$, and thus $t_n < t$ for all $n \in \mathbb{Z}_+$. Again, a similar argument applies.

Conversely, suppose that (2) holds. Then $A.1'$, $A.3.1'$ and $A.4'$ are clearly implied. To check $A.2'$, let $x \in X$, $f \in \mathcal{F}_0$ and $l_1, l_2, l_3 \in \mathcal{L}_1$ be given such that $l_1\lambda l_3 \in \mathcal{L}_1$ for some $\lambda \in (0,1)$. Assume that $(l_1, x) \succ' (f, l_2) \succ' (l_3, x)$. Note that $L_{l_1\lambda l_3}(x) = \lambda L_{l_1}(x) + (1-\lambda)L_{l_3}(x)$ for all $\lambda \in [0,1]$. Hence, there must exist $\alpha, \beta \in (0,1)$ such that $(l_1\alpha l_3, x) \succ' (f, l_2) \succ' (l_1\beta l_3, x)$. Assume that $(f, l_1) \succ' (l_2, x) \succ' (f, l_3)$. If either $(f, l_1) \notin \max \mathcal{M}$ or $(f, l_3) \notin \min \mathcal{M}$, then $v_f(\mathbb{R}) \neq \{0,1\}$, and thus v_f is continuous. Hence, $v_f(\mathbb{R})$ is connected, and then there must exist $T \in \mathbb{R}$ such that $v_f(E_{l_1}u) > v_f(T) > V(l_2, x) > v_f(E_{l_3}u)$. Since v_f is non-increasing, then $T \in (E_{l_1}u, E_{l_3}u)$. Therefore, we can find $\alpha \in (0,1)$ such that $E_{l_1\alpha l_3}u$, which implies that $(f, l_1\alpha l_3) \succ' (l_2, x)$. Next, consider a non-increasing sequence $\{D(f, l_1\frac{1}{n}l_3)\}_{n=1}^{\infty}$ of elements in $\mathcal{D}$. Note that $\cap_{n=1}^{\infty} D(f, l_1\frac{1}{n}l_3) = D(f, l_3)$. Since $c : \mathcal{D} \to \mathbb{R}$ is continuous, then $c(D(f, l_3)) = \lim_{n\to\infty} c(D(f, l_1\frac{1}{n}l_3))$. Moreover, $v_f(E_{l_3}u) = c(D(f, l_3))$, and $c(D(f, l_1\frac{1}{n}l_3))$ weakly decreases in n, so there must exist $N \in \mathbb{Z}_+$ such that $V(l_2, x) > c(D(f, l_1\frac{1}{N}l_3))$. Hence, $(l_2, x) \succ' (f, l_1\frac{1}{N}l_3)$.

□

Proof of Lemma 3. Suppose (1) holds. Since $A.3.2'$ implies $A.3.1'$, then the representation for $\succsim'$, the uniqueness of c and the continuity of v_f follow from Lemma 2. By $A.3.2'(3)$, for each $f \in \mathcal{F}_a$, v_f is strictly decreasing on $v_f^{-1}((0,1))$. We would like to show the rest of (2). For any $f \in \mathcal{F}_0$, let $\overline{x}_f, \underline{x}_f \in X$ be such that $l_{\overline{x}_f} \succsim l_x \succsim l_{\underline{x}_f}$ for all $x \in f(S)$. Let $\lambda_f = \sup\{\lambda \in [0,1] \mid (f, l_{\overline{x}_f}\lambda l_{\underline{x}_f}) \in \max \mathcal{M}\}$. Clearly, λ_f is well-defined since $l_{\underline{x}_f} \in \max \mathcal{M}$. Let $l_f = l_{\overline{x}_f}\lambda_f l_{\underline{x}_f}$. If $\lambda_f = 0$, then $(f, l_f) \in \max \mathcal{M}$. If $\lambda_f > 0$, then there is a non-decreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of real numbers in $[0,1]$ such that $(f, l_{\lambda_n}) \in \max \mathcal{M}$ for each $n \in \mathbb{Z}_+$, and $\lim_{n\to\infty} \lambda_n = \lambda_f$. Thus $\{D(f, l_{\lambda_n})\}_{n=1}^{\infty}$ is a non-increasing sequence of sets in $\mathcal{D}$ such that $D(f, l_f) = \cap_{n=1}^{\infty} D(f, l_{\overline{x}_f}\lambda_n l_{\underline{x}_f})$. By the upper continuity of c, $(f, l_f) \in \max \mathcal{M}$. Let $C = \cap_{f\in\mathcal{F}_0} D(f, l_f)$. Clearly, C is closed and convex since it is the intersection of a family of closed and convex sets.

We introduce some notations which will be useful in the following proof. For any $(f, l) \in \mathcal{F}_0 \times \mathcal{L}_1$, let $\overline{H}(f,l) = \{r \in \mathbb{R}^{|S|} \mid u(f)\cdot r \geq E_l u\}$, $\underline{H}(f,l) = \{r \in \mathbb{R}^{|S|} \mid u(f)\cdot r \leq E_l u\}$, $\overline{H}^{\circ}(f,l) = \{r \in \mathbb{R}^{|S|} \mid u(f)\cdot r > E_l u\}$ and $\underline{H}^{\circ}(f,l) = \{r \in \mathbb{R}^{|S|} \mid u(f)\cdot r < E_l u\}$.

In the following, we show that for all $f \in \mathcal{F}_0$, $E_{l_f}u = \min_{p\in C} E_p u(f)$, and as a result C is non-empty. Fix $f \in \mathcal{F}_0$. Since $E_{l_f}u = \min_{p\in D(f,l_f)} E_p u(f)$ and $C \subseteq D(f, l_f)$, then $E_{l_f}u \leq \min_{p\in C} E_p u(f)$. Suppose the contrary that $E_{l_f}u < \min_{p\in C} E_p u(f)$. Hence, there exists $l' \in \mathcal{L}_1$ such that $E_{l_f}u < E_{l'}u < \min_{p\in C} E_p u(f)$. We can assume without loss of generality that $E_{l'}u = 0$. (Otherwise, consider $\bar{f} \in \mathcal{F}_0$ such that $u(\bar{f}(s)) = u(f(s)) - E_{l'}u$ for all $s \in S$. It is easy to check that $\lambda_f = \lambda_{\bar{f}}$, so $E_{l_f}u < E_{l'}u < \min_{p\in C} E_p u(f)$ if and only if $E_{l_{\bar{f}}}u < 0 < \min_{p\in C} E_p u(\bar{f})$.) Since $E_{l'}u < \min_{p\in C} E_p u(f)$, then either $C = \emptyset$, or for all $p \in C$, $E_p u(f) > E_{l'}u$ and thus $E_p u(-f) < E_{-l'}u$. In both cases, $D(-f, -l') \cap C = \emptyset$. In other words, for each $p \in D(-f, -l')$, there exists $g \in \mathcal{F}_0$ such that $p \notin D(g, l_g)$, or $p \in \underline{H}^{\circ}(g, l)$ for some $l \in \mathcal{L}_1$ satisfying $l_g \succ l$. Since $D(-f, -l')$ is compact, there exist $g_1, ..., g_N \in \mathcal{F}_0$ and $l_1, ..., l_N \in \mathcal{L}_1$ such that $l_{g_n} \succ l_n$ for all $n = 1, ..., N$, and $D(-f, -l') \subseteq \cup_{n=1}^{N} \underline{H}^{\circ}(g_n, l_n)$. Let $x_0, x_1 \in X$ be such that $u(x_0) = 0$ and $u(x_1) = 1$. Let $S = \{s_1, ..., s_{|S|}\}$, and let $h_m \in \mathcal{F}_0$ be such that $h_m(s_m) = x_1$ and $h_m(s) = x_0$ when $s \neq s_m$, $m = 1, ..., |S|$. Note that $\Delta(S) \subseteq \cap_{m=1}^{|S|} \overline{H}(h_m, l_{x_0})$, and thus $D(-f, -l') \subseteq [\cup_{n=1}^{N} \underline{H}^{\circ}(g_n, l_n)] \cap [\cap_{m=1}^{|S|} \overline{H}(h_m, l_{x_0})]$. Again without loss of generality, we assume that $E_{l_n}u = 0$, $n = 1, ..., N$, otherwise we can change $g_1, ..., g_N$ so that this holds while the relations above remain the same. Next we quote Farkas' lemma.

Lemma 5 (Farkas' lemma)**.** *For any $i \times j$ matrix B and i-dimensional vector b, $i, j \in \mathbb{Z}_+$, exactly one of the following two statements is true.*

(i) There exists $q \in \mathbb{R}^j$ such that $Bq = b$ and $b \geq \mathbf{0}$.

(ii) There exists $r \in \mathbb{R}^i$ such that $B^T r \geq \mathbf{0}$ and $b^T r < 0$.

We consider $u(f)$ as a vector in $\mathbb{R}^{|S|}$, i.e., $u(f) = (u(f(s_1)), ..., u(f(s_{|S|})))^T$. Let $b = u(f)$, and similarly let $B = [u(g_1), ..., u(g_N), u(h_1), ..., u(h_{|S|})]$ be a $|S| \times (N + |S|)$ matrix. Suppose that $r \in \mathbb{R}^S$ and $B^T r \geq \mathbf{0}$. That is, $u(g_n)^T r \geq 0$ for all $n = 1, ..., N$, and $u(h_m)^T r \geq 0$, i.e. $r_m \geq 0$, for all $m = 1, ..., |S|$. Thus, $r \in [\cap_{n=1}^{N} \overline{H}(g_n, l_n)] \cap [\cap_{m=1}^{|S|} \overline{H}(h_m, l_{x_0})]$. If $r = \mathbf{0}$, then $b^T r = 0$. If $r \neq \mathbf{0}$, then $\sum_{m=1}^{|S|} r_m > 0$. Since $u(g_n)^T \frac{r}{\sum_{m=1}^{|S|} r_m} \geq 0$ for each $n = 1, ..., N$, then $\frac{r}{\sum_{m=1}^{|S|} r_m} \notin D(-f, -l')$. That is, $u(-f)^T \frac{r}{\sum_{m=1}^{|S|} r_m} < E_{-l'} u$, and thus $u(f)^T \frac{r}{\sum_{m=1}^{|S|} r_m} > E_{l'} u = 0$. Therefore, (ii) does not hold, and there exists $q \geq \mathbf{0}$ in $\mathbb{R}^j$ such that $Bq = b$. If $b = \mathbf{0}$, then $E_{l_f} u = 0$ which contradicts with $E_{l_f} u < E_{l'} u = 0$. Hence, $b \neq \mathbf{0}$, and then $q \neq \mathbf{0}$. Thus, $B \frac{q}{\sum_{n=1}^{N+|S|} q_n} = \frac{b}{\sum_{n=1}^{N+|S|} q_n}$ where $\sum_{n=1}^{N+|S|} q_n > 0$.

For all $n = 1, ..., N$, fix $-g \in \mathcal{F}_0$, $-l_n, -l_{g_n} \in \mathcal{L}_1$ such that $u(-g) = -u(g)$, $E_{-l_n} u = -E_{l_n} u$ and $E_{-l_{g_n}} u = -E_{l_{g_n}} u$. Since $(g_n, l_{g_n}) \in \max \mathcal{M}$ and $-l_n \succ -l_{g_n}$, then $(-g_n, -l_n) \in \min \mathcal{M}$ by $A.5'$, $n = 1, ..., N$. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of elements in X such that $u(x_k) < 0$ for all $k \in \mathbb{Z}_+$ and $\lim_{n \to \infty} u(x_n) = 0$. For all $m = 1, ..., |S|$ and $k \in \mathbb{Z}_+$, fix $-h_m \in \mathcal{F}_0$, $-l_{x_k} \in \mathcal{L}_1$ such that $u(-h_m) = -u(h_m)$ and $E_{-l_{x_k}} u = -E_{l_{x_k}} u$. Since $(h_m, l_{x_0}) \in \max \mathcal{M}$ and $-l_{x_k} \succ l_{x_0}$, then $(-h_m, -l_{x_k}) \in \min \mathcal{M}$ by $A.5'$. By $A.6'$, $(\sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n}(-g_n) + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n}(-h_m), \sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n}(-l_n) + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n}(-l_{x_k})) \in \min \mathcal{M}$, $k \in \mathbb{Z}_+$. By the upper continuity of c, $(\sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n} g_n + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n} h_m, \sum_{n=1}^{N} \frac{q_n}{\sum_{n=1}^{N+|S|} q_n} l_n + \sum_{m=1}^{|S|} \frac{q_{N+m}}{\sum_{n=1}^{N+|S|} q_n} l_{x_k}) \in \max \mathcal{M}$, $k \in \mathbb{Z}_+$. Let $f' \in \mathcal{F}_0$ such that $u(f') = \frac{u(f)}{\sum_{n=1}^{N+|S|} q_n}$. Since $B \frac{q}{\sum_{n=1}^{N+|S|} q_n} = \frac{b}{\sum_{n=1}^{N+|S|} q_n}$ and $b = u(f)$, then $(f', l') \in \max \mathcal{M}$. Since $E_p u(f') \geq E_{l'} u$ if and only if $E_p u(f) \geq E_{l'} u \sum_{n=1}^{N+|S|} q_n = E_{l'} u$, then $(f, l') \in \max \mathcal{M}$. Since $l' \succ l_f$, this contradicts the choice of l_f.

Next we show that c is compatible with the set C. Suppose that $D_1 = D(f_1, l_1)$, $D_2 = D(f_2, l_2)$ and $D_1 \cap C \supseteq D_2 \cap C$. If $D_2 \cap C = \emptyset$, then $E_{l_2} u > \max_{p \in C} E_p u(f_2)$. Fix $-f_2 \in \mathcal{F}_0$, $-l_2 \in \mathcal{L}_1$ such that $u(-f_2) = -u(f_2)$ and $E_{-l_2} u = -E_{l_2} u$. Thus, $E_{-l_2} u < \min_{p \in C} E_p u(-f_2) = E_{l_{-f_2}} u$. Since $(-f_2, l_{-f_2}) \in \max \mathcal{M}$ and $l_{-f_2} \succ -l_2$, then $l_2 \succ -l_{-f_2}$ and thus by $A.5'$, $(f_2, l_2) \in \min \mathcal{M}$. Hence, $c(D_1) \geq 0 = c(D_2)$. If $D_1 \cap C = C$, then $C \subseteq D_1$, that is, $E_{l_1} u \leq \min_{p \in C} E_p u(f_1) = E_{l_{f_1}} u$. Since $(f_1, l_{f_1}) \in \max \mathcal{M}$ and $l_{f_1} \succsim l_1$, then $(f_1, l_1) \in \max \mathcal{M}$ by $A.3.2'$. Hence, $c(D_1) = 1 \geq c(D_2)$.

Suppose that $C \supsetneq D_1 \cap C \supseteq D_2 \cap C \supsetneq \emptyset$. Let $A = \{r \in \mathbb{R}^{|S|} \mid u(f_1)^T r < E_{l_1} u \text{ and } u(f_2)^T r \geq E_{l_2} u\}$. If $A \cap \Delta(S) = \emptyset$, then for each $p \in \Delta(S)$, $(f_2, p) \succsim l_2$ implies that $(f_1, p) \succsim l_1$, i.e., $D_1 \supseteq D_2$. Thus, $c(D_1) \geq c(D_2)$. Suppose that $A \cap \Delta(S) \neq \emptyset$. Thus, $A \neq \emptyset$, and it is easy to see that the interior of A is non-empty. Since $D_1 \cap C \supseteq D_2 \cap C$, then $A \cap C = \emptyset$. Therefore, by a basic separation theorem (Dunford and Schwartz (1966), V.1.12), there exists a non-zero linear functional I on $\mathbb{R}^{|S|}$ and a real number λ such that $I(r) \geq \lambda \geq I(r')$ for all $r \in C$ and $r' \in A$. Let e_n be a vector in $\mathbb{R}^{|S|}$ that takes 1 in the n-th coordinate and 0 in the other coordinates, $n = 1, ..., |S|$. Pick $f_3 \in \mathcal{F}_0$ such that $u(f_3(s_n)) = I(e_n)$, $n = 1, ..., |S|$. Then $I(r) = u(f_3)^T r$ for each $r \in \mathbb{R}^{|S|}$. Pick $l_3 \in \mathcal{L}_1$ such that $E_{l_3}u = \lambda$. Since $I(C) \geq \lambda$, then $\min_{p \in C} E_p u(f_3) \geq E_{l_3}u$, and thus $(f_3, l_3) \in \max \mathcal{M}$.

If $(f_2, l_2) \in \max \mathcal{M}$, then $C \subseteq D_2$ by the construction of C. Thus $C = D_2 \cap C$ which contradicts our assumption. Hence, either $v_{f_2}(E_{l_2}u) \in (0, 1)$ or $(f_2, l_2) \in \min \mathcal{M}$. In both cases, there exists a sequence $\{l'_n\}_{n=1}^{\infty}$ of elements in $\mathcal{L}_1$ such that $l'_n \succ l_2$ for all $n \in \mathbb{Z}_+$, and $\lim_{n\to\infty} v_{f_2}(E_{l'_n}u) = v_{f_2}(E_{l_2}u)$. Fix $n \in \mathbb{Z}_+$. Fix $p \in \Delta(S)$ such that $(f_3, p) \succsim l_3$ and $(f_2, p) \succsim l'_n$, i.e., $E_p u(f_3) \geq \lambda$ and $E_p u(f_2) \geq E_{l'_n}u > E_{l_2}u$. If $p \in A$, then p is an interior point of A, and thus $E_p u(f_3) < \lambda$ which is a contradiction. Hence, $p \notin A$, and thus $(f_1, p) \succsim l_1$. By $A.3.2'$, $(f_1, l_1) \succsim' (f_2, l'_n)$. Taking the limit, we get that $c(D_1) \geq c(D_2)$.

Let $(f, l) \in \mathcal{M}$ be given. Suppose $\max\{(f, p) \mid p \in C\} \succ l$ and $(f, l) \in \min \mathcal{M}$. Then by $A.5'$ and the upper continuity of c, $(-f, -l) \in \max \mathcal{M}$. Note that $E_{-l}u > \min_{p \in C} E_p u(-f) = E_{l_{-f}}u$. Thus, $-l \succ l_{-f}$ which is a contradiction to the construction of l_{-f}. Hence, $(f, l) \notin \min \mathcal{M}$ and $c(D(f, l)) > 0$. Suppose $l \succ \min\{(f, p) \mid p \in C\}$. Since $E_{l_f}u = \min_{p \in C} E_p u(f)$, then by the choice of l_f, $(f, l) \notin \max \mathcal{M}$. As a result, $c(D(f, l)) < 1$.

To check the uniqueness of C, suppose the contrary that there exists $C' \subseteq \Delta(S)$ satisfying the desirable properties. Suppose that $q \in C' \setminus C$. Then there exists $f \in \mathcal{F}_0$ such that $q \notin D(f, l_f)$, and thus $\min_{p \in C'} E_p u(f) \leq E_q u(f) < E_{l_f}u$. Since c is compatible with C', then $c(D(f, l_f)) < 1$ which is a contradiction. Suppose that $q \in C \setminus C'$. By a separation theorem (Dunford and Schwartz (1966), V.2.10), there exists a linear functional I on $\mathbb{R}^{|S|}$ such that $I(q) < \min_{p \in C'} I(p)$. Pick $f \in \mathcal{F}_0$ such that $u(f(s_n)) = I(e_n)$, $n = 1, ..., |S|$. Hence, for all $p \in \Delta(S)$, $I(p) = E_p u(f)$. Since $q \in C$, then $\min_{p \in C} E_p u(f) \leq E_q u(f) < \min_{p \in C'} E_p u(f)$. Let $l \in \mathcal{L}_1$ be such that $\min_{p \in C} E_p u(f) < E_l u < \min_{p \in C'} E_p u(f)$. Since c is compatible with C, then $c(D(f, l)) < 1$. Since c is compatible with C' and $D(f, l) \cap C' = C'$, then $c(D(f, l)) = 1$, which is a contradiction as desired.

Next, we show that $\mathcal{U} = \{A \subseteq S \mid p(A) = p'(A) \text{ for all } p, p' \in C\}$, which implies that $\mathcal{U}$ is a λ-system. Suppose that $A \in \mathcal{U}$. Let $x_0, x_1, x'_1 \in X$ be given such that $u(x_0) = 0$, $u(x_1) = 1$ and $u(x'_1) = -1$. Since A is unambiguous, then $(x_1 A x_0, l_{x_1} \lambda l_{x_0}) \in \min \mathcal{M}$ when $\lambda > \lambda_{x_1 A x_0}$. Thus, $(x'_1 A x_0, l_{x'_1} \lambda l_{x_0}) \in \max \mathcal{M}$ when $\lambda > \lambda_{x_1 A x_0}$. By the upper continuity of c, $(x'_1 A x_0, l_{x'_1} \lambda_{x_1 A x_0} l_{x_0}) \in \max \mathcal{M}$. Hence, $\lambda_{x'_1 A x_0} \geq 1 - \lambda_{x_1 A x_0}$. Suppose $\lambda_{x'_1 A x_0} > 1 - \lambda_{x_1 A x_0}$. Then $l_{x_0} \lambda_{x'_1 A x_0} l_{x'_1} \succ l_{x'_1} \lambda_{x_1 A x_0} l_{x_0}$. Since $(x_1 A x_0, l_{x_1} \lambda_{x_1 A x_0} l_{x_0}) \in \max \mathcal{M}$, then by $A.5'$, $(x'_1 A x_0, l_{x_0} \lambda_{x'_1 A x_0} l_{x'_1}) \in \min \mathcal{M}$, which contradicts the choice of $\lambda_{x'_1 A x_0}$. Hence, $\lambda_{x'_1 A x_0} = 1 - \lambda_{x_1 A x_0}$. For any $p \in C$, $p \in D(x_1 A x_0, l_{x_1 A x_0}) \cap D(x'_1 A x_0, l_{x'_1 A x_0})$. That is, $p(A) \geq \lambda_{x_1 A x_0}$ and

$-p(A) \geq -(1-\lambda_{x_1' A x_0}) = -\lambda_{x_1 A x_0}$. Hence, $p(A) = \lambda_{x_1 A x_0}$ for all $p \in C$.

On the other hand, suppose that $A \subseteq S$ and $p(A) = p'(A)$ for all $p, p' \in C$. Let $x, y \in X$ be given. Note that $\max_{p \in C} E_p u(xAy) = \min_{p \in C} E_p u(xAy) = E_{l_{xAy}} u$. Fix $l \in \mathcal{L}_1$. If $l_{xAy} \succsim l$, then $(xAy, l) \in \max \mathcal{M}$. If $l \succ l_{xAy}$, then $D(xAy, l) \cap C = \emptyset$, and thus $c(D(xAy, l)) = 0$, i.e., $(xAy, l) \in \min \mathcal{M}$. Hence, $A \in \mathcal{U}$.

Conversely, suppose that (2) holds. By Lemma 2, $A.1'$, $A.2'$, $A.3.2'(1)$ and $A.4'$ holds. To see $A.3.2'(2)$, note that $(f_3, l_3) \in \max \mathcal{M}$ implies that $\min\{(f_3, p) \mid p \in C\} \succsim l_3$, and thus $C \subseteq D(f_3, l_3)$. Hence, if $(f_2, p) \succsim l_2$ implies $(f_1, p) \succsim l_1$ either for each $p \in \Delta(S)$ or for each $p \in D(f_3, l_3)$ where $(f_3, l_3) \in \max \mathcal{M}$, then $D(f_1, l_1) \cap C \supseteq D(f_2, l_2) \cap C$, so that $(f_1, l_1) \succsim' (f_2, l_2)$. $A.3.2'(3)$ follows from that v_f is strictly decreasing on $v_f^{-1}((0,1))$ if $f \in \mathcal{F}_a$.

Finally, we check $A.5'$. Suppose $(f, l) \in \max \mathcal{M}$. Then $C \subseteq D(f, l)$. If $l' \succ -l$, then $D(-f, l') \cap D(f, l) = \emptyset$, and thus $D(-f, l') \cap C = \emptyset$. Hence, $D(-f, l') \in \min \mathcal{M}$. Suppose $(f, l) \in \min \mathcal{M}$. Then $l \succsim \max\{(f, p) \mid p \in C\}$, and thus $\min_{p \in C} E_p u(-f) \geq E_{-l} u$. It follows that $C \subseteq D(-f, -l) \subseteq D(-f, l')$ if $-l \succ l'$. Hence, $(-f, l') \in \max \mathcal{M}$. Suppose that $(f_1, l_1), (f_2, l_2) \in \min \mathcal{M}$ and $\lambda \in (0,1)$. Then $E_{l_1} \geq \max_{p \in C} E_p u(f_1)$ and $E_{l_2} \geq \max_{p \in C} E_p u(f_2)$. For any $p \in C$, $E_p u(f_1 \lambda f_2) = \lambda E_p u(f_1) + (1-\lambda) E_p u(f_2) \leq \lambda E_{l_1} u + (1-\lambda) E_{l_2} u = E_{l_1 \lambda l_2} u$. If $\max_{p \in C} E_p u(f_1 \lambda f_2) < E_{l_1 \lambda l_2} u$, then $D(f_1 \lambda f_2, l_1 \lambda l_2) \cap C = \emptyset$, and thus $(f_1 \lambda f_2, l_1 \lambda l_2) \in \min \mathcal{M}$. If $\max_{p \in C} E_p u(f_1 \lambda f_2) = E_{l_1 \lambda l_2} u$, then $E_{l_n} u = \max_{p \in C} E_p u(f_n)$ for each $n = 1, 2$. Hence, $\max_{p \in C} E_p u(f_i) > \min_{p \in C} E_p u(f_n)$ for each $n = 1, 2$, otherwise $(f_1, l_1), (f_2, l_2) \in \max \mathcal{M}$. Let $q \in \arg\min_{p \in C} E_p u(f_1)$. Then $\min_{p \in C} E_p u(f_1 \lambda f_2) \leq E_q u(f_1 \lambda f_2) < \lambda \max_{p \in C} E_p u(f_1) + (1-\lambda) \max_{p \in C} E_p u(f_2) = E_{l_1 \lambda l_2} u$. Since $\min_{p \in C} E_p u(f_1 \lambda f_2) < E_{l_1 \lambda l_2} u = \max_{p \in C} E_p u(f_1 \lambda f_2)$, then $f_1 \lambda f_2 \in \mathcal{F}_a$. By the continuity of $v_{f_1 \lambda f_2}$, $v_{f_1 \lambda f_2}(E_{l_1 \lambda l_2} u) = 0$, i.e., $(f_1 \lambda f_2, l_1 \lambda l_2) \in \min \mathcal{M}$.

□

Proof of Theorem 1. We only check the preference representation result. Suppose (1) holds. We gradually prove the results by establishing the following facts.

F.1. For all $f \in \mathcal{F}_0$ and $l \in \mathcal{L}_1$ such that $l_{f(s)} \sim l$ for each $s \in S$, then $f \sim l$.

Pick $x \in X$ such that $l_x \sim l$. By $A.2.2$, $f_x \sim l_x$ and thus $f_x \sim l$. Since $(f, l) \sim' (f_x, l)$ for all $l \in \mathcal{L}_1$, then by A.6, $f \sim f_x \sim l$.

F.2. For all $f, g \in \mathcal{F}_0$, if $l_{f(s)} \succsim l_{g(s)}$ for all $s \in S$, then $f \succsim g$.

For any $l \in \mathcal{L}_1$ and $p \in \Delta(S)$, $(g, p) \succsim l$ implies that $(f, p) \succsim l$, and thus $(f, l) \succsim' (g, l)$ by $A.3.2'(2)$. Hence, $f \succsim g$ by $A.6$.

F.3. For any $f \in \mathcal{F}_0$, there exists $l \in \mathcal{L}_1$ such that $f \sim l$. Moreover, if $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, then $f \sim \underline{l}_f$.

Let $f_0, f_1 \in \mathcal{F}_0$ be such that for all $s \in S$, $l_{f_0(s)} \sim \underline{l}_f$ and $l_{f_1(s)} \sim \bar{l}_f$. By $A.6$, $f_1 \succsim f \succsim f_0$. If $f \sim f_0$ or $f \sim f_1$, then $f \sim \underline{l}_f$ or $f \sim \bar{l}_f$ by F.1. Suppose that $f_1 \succ f \succ f_0$. By $A.3.2$, there exist $l', l'' \in \mathcal{L}_1$ such that $f_1 \succ l' \succ f \succ l'' \succ f_0$. Thus, $\bar{l}_f \succ l' \succ f \succ l'' \succ \underline{l}_f$. One can check that there exists $\gamma \in (0, 1)$ such that $\bar{l}_f \lambda \underline{l}_f \succ f$ if $\lambda \in (\gamma, 1]$ and $f \succ \bar{l}_f \lambda \underline{l}_f$ if $\lambda \in [0, \gamma)$. If $f \succ \bar{l}_f \gamma \underline{l}_f$, then by F.1, $f \succ f_1 \gamma f_0$. By $A.3.2$, there exists $l''' \in \mathcal{L}_1$ such that $f \succ l''' \succ f_1 \gamma f_0$. Thus, $\bar{l}_f \succ f \succ l''' \succ \bar{l}_f \gamma \underline{l}_f$, and then $l''' \sim \bar{l}_f \lambda \underline{l}_f$ for some $\lambda \in (\gamma, 1)$ which is a contradiction to the construction of γ. Similarly, it is not true that $\bar{l}_f \gamma \underline{l}_f \succ f$. Hence, $f \sim \bar{l}_f \gamma \underline{l}_f$, as desired.

If $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, then $\underline{l}_f \sim \bar{l}_f$ and thus $f_1 \sim f_0$. Hence, $f \sim f_0 \sim \underline{l}_f$.

Define $I : \mathbb{R}^{|S|} \to \mathbb{R}$ by $I(r) = E_l u$ if where $r \in \mathbb{R}^{|S|}$, $f \in \mathcal{F}_0$ and $l \in \mathcal{L}_1$ satisfy $r = u(f)$ and $f \sim l$. By $F.2$ and $F.3$, I is well-defined. Let $e \in \mathbb{R}^{|S|}$ be the unit vector, i.e., $e_n = 1$ for all $n = 1, ..., |S|$. It is easy to see that (1) $I(u(f)) \geq I(u(g))$ if and only if $f \succsim g$, (2) $I(te) = t$ for all $t \in \mathbb{R}$, and (3) $I(r) \geq I(r')$ for all $r \geq r'$ in $\mathbb{R}^{|S|}$. For any $f \in \mathcal{F}_0$ and $t \in \mathbb{R}$, let f_t denote an act in $\mathcal{F}_0$ such that $u(f_t) = u(f) + te$. Note that for all $f \in \mathcal{F}_0, t, k \in \mathbb{R}$, $v_f(k) = v_{f_t}(k+t)$, and thus, $E_{\underline{l}_{f_t}} u = E_{\underline{l}} u + t$ and $E_{\bar{l}_{f_t}} u = E_{\bar{l}} u + t$.

F.4. There exist $k, k' > 0$ such that $I(u(f) + te) - I(u(f)) \in [k't, kt]$ for all $t > 0$ and $f \in \mathcal{F}_a$.

Suppose $x_0 \in X$ and $u(x_0) = 0$. Let $f \in \mathcal{F}_a$ and $t > 0$ be given. Let $\alpha, \beta \in (0, 1)$ be given as in $A.4.2$. Suppose $x_f \in X$ and $f \sim l_{x_f}$. Note that x_f always exists. Then $I(u(f)) = I(u(x_f)e) \to I(\alpha \frac{u(f)}{\alpha} + (1-\alpha)\mathbf{0}) = I(\beta \frac{u(x_f)e}{\beta} + (1-\beta)\mathbf{0}) \to I(u(\alpha f' + (1-\alpha) f_{x_0})) = I(u(\beta g' + (1-\beta) f_{x_0}))$ where $f', g' \in \mathcal{F}_0$ satisfy $u(f') = \frac{u(f)}{\alpha}$ and $u(g') = \frac{u(x_f)e}{\beta}$. Hence, $\alpha f' + (1-\alpha) f_{x_0} \sim \beta g' + (1-\beta) f_{x_0}$. Let $y \in X$ and $u(y) = \frac{t}{1-\alpha}$. Since $g' \in \mathcal{F}_0 \setminus \mathcal{F}_a$ and $y \succ x_0$, then $\beta \frac{g'}{\beta} + (1-\beta) f_y \succsim \alpha f' + (1-\alpha) f_y$. Thus, $\beta \frac{u(x_f)}{\beta} + (1-\beta)0 + (1-\beta)\frac{t}{1-\alpha} \geq I(u(\alpha f' + (1-\alpha) f_y)) \to I(u(\beta g' + (1-\beta) f_{x_0})) + \frac{1-\beta}{1-\alpha} t \geq I(u(f) + te) \to I(u(\alpha f' + (1-\alpha) f_{x_0})) + \frac{1-\beta}{1-\alpha} t \geq I(u(f) + te) \to I(u(f) + te) - I(u(f)) \leq \frac{1-\beta}{1-\alpha} t$. Similarly, $I(u(f) + te) - I(u(f)) \geq \frac{1-\gamma}{1-\alpha} t$. Hence, $k' = \frac{1-\gamma}{1-\alpha}$ and $k = \frac{1-\beta}{1-\alpha}$.

F.5. For any $f, g \in \mathcal{F}_0$, if $f(s) \succ g(s)$ for all $s \in S$, then $f \succ g$.

Let $t = \min_{s \in S}[u(f(s)) - u(g(s))]$. Note that $t > 0$ and $u(g) \geq u(f) + te$. By $F.2$, it suffices to show that $I(u(f) + te) > I(u(f))$, which follows from $F.3$ and $F.4$.

F.6. For all $x \in X$ and $f \in \mathcal{F}_a$, there exists $t \in \mathbb{R}$ such that $f_t \sim f_x$.

Fix $x \in X$ and $f \in \mathcal{F}_a$. Pick $t_0, t_1 \in \mathbb{R}$ such that $\min_{s \in S} u(f_{t_1}(s)) \geq u(x) \geq \max_{s \in S} u(f_{t_0})$. By $F.2$, $f_{t_1} \succsim f_x \succsim f_{t_0}$. If $f_{t_1} \sim f_x$ or $f_{t_0} \sim f_x$, then by $A.2.2$, we are done. Suppose that $f_{t_1} \succ f_x \succ f_{t_0}$. Thus, $t_1 > t_0$ by $F.2$. Fix $f_{t_1} \lambda f_{t_0} \in \mathcal{F}$ where $\lambda \in [0, 1]$. Note that $u(f_{t_1} \lambda f_{t_0}) = u(f_{t_0}) + \lambda(t_1 - t_0)e$ and $f_{t_0} \in \mathcal{F}_a$. By $F.4$, $I(u(f_{t_1} \lambda f_{t_0})) \leq I(u(f_{t_0})) + k\lambda(t_1 - t_0)$ for some $k \in \mathbb{R}$. Hence, there exists $\lambda \in (0, 1)$ such that $f_x \succ f_{t_1} \lambda f_{t_0}$. Similarly, there exists $\lambda' \in (0, 1)$ such that $f_{t_1} \lambda f_{t_0} \succ f_x$. Hence, we can find $\gamma \in (0, 1)$ such that $f_x \succ f_{t_1} \lambda f_{t_0}$ for all $\lambda \in [0, \gamma)$ and $f_{t_1} \lambda f_{t_0} \succ f_x$ for all $\lambda \in (\gamma, 1]$. Suppose that $f_x \succ f_{t_1} \gamma f_{t_0}$. When $\lambda \in (\gamma, 1]$, $u(f_{t_1} \lambda f_{t_0}) = u(f_{t_1} \gamma f_{t_0}) + (\lambda - \gamma)(t_1 - t_0)$, and then $I(u(f_{t_1} \lambda f_{t_0})) \leq I(u(f_{t_1} \gamma f_{t_0})) + k(\lambda - \gamma)(t_1 - t_0)$. Hence, there exists $\lambda \in (\gamma, 1]$ such that $f_x \succ f_{t_1} \lambda f_{t_0} \succ f_{t_1} \gamma f_{t_0}$, which is a contradiction to the construction of γ. Similarly, it is not true that $f_{t_1} \gamma f_{t_0} \succ f_x$. Hence, $f_{\gamma t_1 + (1-\gamma) t_0} \sim f_{t_1} \gamma f_{t_0} \sim f_x$.

Suppose that $\mathcal{F}_a \neq \emptyset$. For each $x \in X$, let $F_x = \{f \in \mathcal{F}_a \mid f \sim f_x\}$ and $U_x = \cap_{f \in F_x} \{r \in \mathbb{R} \times [0, 1] \mid r_1 \geq E_{\underline{l}_f} u, r_2 \geq v_f(r_1)\}$. Note that $F_x \neq \emptyset$. Moreover, U_x is closed in $\mathbb{R} \times [0, 1]$ since v_f is continuous when $f \in \mathcal{F}_a$.

F.7. Suppose that $\mathcal{F}_a \neq \emptyset$. For any $x, y \in X$, if $f_y \succ f_x$, then $U_x \supseteq U_y$.

Suppose that $r \in U_y$ and $f \in F_x$. Then there exists $t \in \mathbb{R}$ such that $f_t \sim f_y$. Hence, $f_t \in F_y$. Since $r \in U_y$ and $f_t \sim f_y \succ f_x \sim f$, then $r_1 \geq E_{\underline{l}_{f_t}} u \geq E_{\underline{l}_f} u$ and $r_2 \geq v_{f_t}(r_1) = v_f(r_1 - t) \geq v_f(r_1)$. Hence, $r \in U_x$.

F.8. Suppose that $\mathcal{F}_a \neq \emptyset$. For any $x \in X$, $(u(x), 1) \in U_x$, and thus $U_x \neq \emptyset$.

Fix $f \in F_x$. It suffices to show that $u(x) \geq E_{\underline{l}_f} u$. Suppose the contrary that $u(x) < E_{\underline{l}_f} u$. There exists $t < 0$ such that $u(x) < E_{\underline{l}_{f_t}} u$. By $A.6$, $f_t \succsim f_x$. Hence, $f \succ f_t \succsim f_x$ which is a contradiction to $f \sim f_x$.

F.9. Suppose that $\mathcal{F}_a \neq \emptyset$. Let $x, y \in X$ be given. If $f_y \succ f_x$, then $(u(x), k) \notin U_y$ for all $k \in [0, 1]$.

Since $f_y \succ f_x$, then there must exist $f \in \mathcal{F}_a$ such that $u(y) > E_{\bar{l}_f} u > E_{\underline{l}_f} u > u(x)$. Suppose that $f \sim f_z$. Thus, $(u(x), k) \notin U_z$ for all $k \in [0, 1]$, and $f_y \succ f_z$. By F.7, $(u(x), k) \notin U_y$ for all $k \in [0, 1]$.

F.10. Let $x, y \in X$ be given so that $f_y \succ f_x$. Suppose that $f \in F_x$ and $(k, v_f(k)) \in U_x$ where $k \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]$. Then $(k, v_f(k)) \notin U_y$.

Since $f_y \succ f_x \sim f$, then there exists $t > 0$ such that $f_t \sim f_y$. If $k = E_{\underline{l}_f} u$, then $k < E_{\underline{l}_{f_t}} u$, and thus $(k, v_f(k)) \notin U_y$. If $E_{\underline{l}_f} u < k \leq E_{\bar{l}_f} u$, then $v_{f_t}(k) = v_f(k - t) > v_f(k)$ by the strict-increasing property of v_f. Hence, $(k, v_f(k)) \notin U_y$.

F.11. Suppose that $\mathcal{F}_a \neq \emptyset$ and $(r_1, r_2) \in U_x$ for some $x \in X$. If $r_1' \geq r_1$, then $(r_1', r_2) \in U_x$. If there exists $r_1' < r_1$ such that $(r_1', r_2) \in U_x$, then there exists $y \in X$ such that $f_y \succ f_x$ and $(r_1, r_2) \in U_y$.

If $r_1' \geq r_1$, then for any $f \in F_x$, $r_1' \geq r_1 \geq E_{\underline{l}_f} u$ and $r_2 \geq v_f(r_1) \geq v_f(r_1')$. Hence, $(r_1', r_2) \in U_x$.

Suppose that there exists $r_1' < r_1$ such that $(r_1', r_2) \in U_x$. Let $R = r_1 - r_1'$. Let $k' > 0$ be given as in *F*.4. By F.9, $r_1 > r_1' \geq u(x)$. Suppose the contrary that for all $y \in X$ such that $f_y \succ f_x$, $(r_1, r_2) \notin U_y$. Pick $y^* \in X$ such that $u(y^*) \in (u(x), r_1]$ and $u(y^*) - u(x) < k'R$. There exists $f \in F_{y^*}$ such that $r_2 < v_f(r_1)$, since $r_1 \geq u(y^*) \geq E_{\underline{l}_f} u$. By *F*.4, $I(u(f)) - I(u(f_{-R})) \geq k'R$. Then $I(u(f_{-R})) \leq I(u(f)) - k'R = u(y^*) - k'R < u(x)$. Hence, by *F*.5 and *F*.6, there exists $\triangle R > 0$ such that $f_{-R+\triangle R} \sim f_x$. Thus, $v_{f_{-R+\triangle R}}(r_1') \geq v_{f_{-R+\triangle R}}(r_1' + \triangle R) = v_f(r_1) > r_2$, which is a contradiction to $(r_1', r_2) \in U_x$.

F.12. Suppose that $\mathcal{F}_a \neq \emptyset$ and $(r_1, r_2) \notin U_x$ for all $x \in X$. Then for any $r_1' \in \mathbb{R}$, $(r_1', r_2) \notin U_x$ for all $x \in X$.

If $r_1' \leq r_1$, then by *F*.11, $(r_1', r_2) \notin U_x$ for all $x \in X$. Suppose that $r_1' > r_1$. Let $R = r_1' - r_1$.

Let $k > 0$ be given as in $F.4$. Suppose the contrary that $r_1' > r_1$ and $(r_1', r_2) \in U_y$ for some $y \in X$. Pick $x^* \in X$ such that $u(x^*) < \min\{r_1, u(y) - kR\}$. Since $(r_1, r_2) \notin U_{x^*}$ and $u(x^*) < r_1$, then there exists $f \in \mathcal{F}_{x^*}$ such that $v_f(r_1) > r_2$. Note that $I(u(f_R)) \leq I(u(f)) + kR$. Thus, $I(u(f_R)) \leq u(x^*) + kR < u(y)$. There exists $\triangle R > 0$ such that $f_{R+\triangle R} \sim f_y$. Hence, $v_{f_{R+\triangle R}}(r_1') = v_{(f_R}(r_1' - \triangle R) \geq v_{f_R}(r_1') = v_f(r_1) > r_2$, which is a contradiction to $(r_1', r_2) \in U_y$, as desired.

Define $w : \mathbb{R} \times [0, 1] \to [-\infty, \infty)$ by $w(r_1, r_2) = r_1$ if $\mathcal{F}_a = \emptyset$ and $w(r_1, r_2) = \sup\{u(x) \mid x \in X \text{ and } (r_1, r_2) \in U_x\}$ if $\mathcal{F}_a \neq \emptyset$.

F.13. The function w is well-define and normalized.

The result is obvious when $\mathcal{F}_a = \emptyset$. Suppose that $\mathcal{F}_a \neq \emptyset$. To see w is well-defined, just note that by $F.9$, $\{u(x) \mid x \in X \text{ and } (r_1, r_2) \in U_x\}$ is bounded above for any $(r_1, r_2) \in \mathbb{R}\times[0, 1]$. The fact that w is normalized follows from $F.8$ and $F.9$.

F.14. Suppose that $\mathcal{F}_a \neq \emptyset$ and $w(r_1, r_2) = u(x)$ for some $x \in X$. Then $(r_1, r_2) \in U_x$.

Suppose the contrary that $(r_1, r_2) \notin U_x$. Hence, there exists $f \in F_x$ such that either $r_1 < E_{\underline{l}_f} u$ or $r_2 < v_f(r_1)$. If $r_1 < E_{\underline{l}_f} u$, then pick $t \in (0, E_{\underline{l}_f} u - r_1)$ so that $r_1 < E_{\underline{l}_{f_{-t}}} u$. Thus, $(r_1, r_2) \notin U_y$ if $f_y \succsim f_{-t}$. Hence, $w(r_1, r_2) \leq I(u(f_{-t})) < I(u(f)) = u(x)$, which is a contradiction. If $r_2 < v_f(r_1)$, then pick $t \in (0, \min\{v_f^{-1}(r_2)\})$ so that $r_2 < v_{f_{-t}}(r_1)$ and thus $(r_1, r_2) \notin U_y$ when $f_y \succsim f_{-t}$. This leads to the same contradiction.

F.15. The function w is upper semicontinuous.

The case when $\mathcal{F}_a = \emptyset$ is clear. Suppose that $\mathcal{F}_a \neq \emptyset$. Note that U_x is a closed set for any $x \in X$. Thus, it suffices to show that for each $t \in \mathbb{R}$, $w^{-1}([t, \infty)) = U_x$ if $u(x) = t$. Let $t \in \mathbb{R}$ and $x \in X$ be given such that $u(x) = t$. For all $(r_1, r_2) \in U_x$, $w(r_1, r_2) \geq u(x) = t$. On the other hand, suppose that $w(r_1, r_2) \geq t$. Suppose also that $w(r_1, r_2) = u(y)$, $y \in X$. Hence, $f_y \succsim f_x$ and $(r_1, r_2) \in U_y$. Thus, $(r_1, r_2) \in U_x$.

F.16. If $w(r_1, r_2) > -\infty$, $r_1' > r_1$ and $r_2' > r_2$, then $w(r_1', r_2) > w(r_1, r_2)$ and

$w(r_1, r_2') \geq w(r_1, r_2)$. If $w(r_1, r_2) = -\infty$, then $w(r_1', r_2) = -\infty$ for all $r_1' \in \mathbb{R}$.

The case when $\mathcal{F}_a = \emptyset$ is easy. Suppose that $\mathcal{F}_a \neq \emptyset$. Let $r_1' > r_1$ and $r_2' > r_2$ be given. Since $w(r_1, r_2) > -\infty$, then $w(r_1, r_2) = u(x)$ for some $x \in X$. Then $(r_1, r_2) \in U_x$ and $(r_1, r_2) \notin U_y$ for any $y \in X$ such that $f_y \succ f_x$. By *F*.11, $(r_1', r_2) \in U_x$, and there exists $y \in X$ such that $f_y \succ f_x$ and $(r_1', r_2) \in U_y$. Thus, $w(r_1', r_2) > w(r_1, r_2)$. The fact that $w(r_1, r_2') \geq w(r_1, r_2)$ is obvious. If $w(r_1, r_2) = -\infty$, then $(r_1, r_2) \notin U_x$ for all $x \in X$. By *F*.12, $w(r_1', r_2) = -\infty$ for all $r_1 \in \mathbb{R}$.

F.17. For any $x \in X$ and $f \in F_x$, there exists $t \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$ such that $(t, v_f(t)) \in U_x$.

Fix $x \in X$ and $f \in F_x$. For all $g \in \mathcal{F}_a$ Denote by U_g the set $\{(r_1, r_2) \in \mathbb{R} \times [0, 1] \mid r_1 \geq E_{\underline{l}_g} u, r_2 \geq v_f(r_1)\}$. Suppose the contrary that for all $t \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$, $(t, v_f(t)) \notin U_x$, i.e., $(t, v_f(t)) \in \mathbb{R} \times [0, 1] \setminus U_g$ for some $g \in F_x$. Note that $G_f := \{(t, v_f(t) \mid t \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]\}$ is compact and $\mathbb{R} \times [0, 1] \setminus U_g$ is open for all $g \in F_x$. Then there exist $g_1, ..., g_N \in F_x$ such that $G_f \subseteq \cup_{n=1}^{N} (\mathbb{R} \times [0, 1] \setminus U_{g_n})$. Since $v_f(E_{\underline{l}_f} u) = 1$, then $\underline{l}_{g_n} \succ \underline{l}_f$ for some $n \in \{1, ..., N\}$. For any $l \in \mathcal{L}_1$ such that $\bar{l}_f \succsim l \succ \underline{l}_f$, $v_f(E_l u) < 1$, and either $E_l u < E_{\underline{l}_{g_n}} u$ or $v_f(E_l u) < v_{g_n}(E_l u)$ for some $n \in \{1, ..., N\}$. In the former case, we also get that $v_f(E_l u) < 1 = v_{f_n}(E_l u)$. Hence, $(f_n, l) \succ' (f, l)$. By A.6, $\max\{g_n \mid n = 1, ..., N\} \succ f$, which is a contradiction.

F.18. The function W defined in Theorem 1 is well-defined and represents $\succsim$. Moreover, W is bounded in translation.

Suppose that $\mathcal{F}_a = \emptyset$. Fix $f \in \mathcal{F}_0$. Then $\underline{l}_f \sim \bar{l}_f$ and $W(f) = w(E_{\underline{l}_f} u, 1) = E_{\underline{l}_f} u$. Moreover, by *F*.3, $f \sim \underline{l}_f$. Clearly, W is well-defined and represents $\succsim$. Suppose that $\mathcal{F}_a \neq \emptyset$. If $f \in \mathcal{F}_0 \setminus \mathcal{F}_a$, then $f \sim \bar{l}_f$ and $W(f) = E_{\underline{l}_f} u$ by F.8 and F.9. If $f \in \mathcal{F}_a$, then by F.3 and F.10, there exists $x \in X$ such that $f \in F_x$. By *F*.17, there exists $t \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$ such that $(t, v_f(t)) \in U_x$, and $(t, v_f(t)) \notin U_y$ for all $y \in X$ such that $f_y \succ f_x$. Hence, $w(t, v_f(t)) = u(x)$. For any $t' \in [E_{\underline{l}_f} u, E_{\bar{l}} u]$ such that $(t', v_f(t')) \notin U_x$, F.7 implies that $(t', v_f(t')) \notin U_y$. By F.14, $w(t', v_f(t')) < u(x)$. Hence, $W(f) = w(t, v_f(t)) = u(x)$, which implies that W is well-defined and represents $\succsim$. Lastly, since $W(f) = I(u(f))$, W is bounded in translation by F.4.

F.19. If $w' : \mathbb{R} \times [0, 1] \to [-\infty, \infty)$ also satisfies the desired properties, then $w' \leq w$.

Suppose that contrary that $w(r_1, r_2) < w'(r_1, r_2)$ for some $(r_1, r_2) \in \mathbb{R} \times [0,1]$. Let $x \in X$ be such that $u(x) = w'(r_1, r_2)$. If $\mathcal{F}_a = \emptyset$, then $u(x) > w(r_1, r_2) = r_1$. Since w' is normalized and strictly increasing, then $r_2 < 1$ and $w'(u(x), 1) = u(x) > w'(r_1, r_2)$, which is a contradiction. If $\mathcal{F}_a \neq \emptyset$, then $(r_1, r_2) \notin U_x$, and thus there exists $f \in \mathcal{F}_x$ such that either $r_1 < E_{\underline{l}_f} u$ or $r_2 < v_f(r_1)$. In either case, there exists $t < 0$ such that $v_{f_t}(r_1) = r_2$ and $r_1 \in [E_{\underline{l}_{f_1}} u, E_{\bar{l}_{f_1}} u]$. Let $W' : \mathcal{L}_1 \cup \mathcal{F}_0 \to \mathbb{R}$ be the corresponding function defined as in Theorem 1 by w'. Thus, $W'(f_t) \geq w'(r_1, r_2) = u(x)$. Hence, $f_t \succsim f_x$, which contradicts that $f_x \sim f \succ f_t$.

Conversely, suppose (2) holds. We only check A.4.2 and A.6.

First, we check A.4.2. Since W is bounded in translation, then there exists $k, k > 0$ such that for any $f \in \mathcal{F}_a$ and $t > 0$, $W(f_t) - W(f) \in [k't, kt]$. Pick $\alpha, \beta, \gamma \in (0,1)$ such that $k' = \frac{1-\gamma}{1-\alpha}$ and $k = \frac{1-\beta}{1-\alpha}$. Fix $f \in \mathcal{F}_a$, $g \in \mathcal{F}_0 \setminus \mathcal{F}_a$ and $x, y \in X$ such that $f_y \succ f_x$.

Then $\beta g + (1-\beta) f_x \succsim \alpha f + (1-\alpha) f_x \to I(u(\beta g + (1-\beta) f_x)) \geq I(u(\alpha f + (1-\alpha) f_x)) \to \beta E_{\underline{l}_g} u + (1-\beta) u(x) \geq I(\alpha u(f) + (1-\alpha) u(x) e) \to I(u(\beta g + (1-\beta) f_y)) = \beta E_{\underline{l}_g} u + (1-\beta) u(y) = \beta E_{\underline{l}_g} u + (1-\beta) u(x) + (1-\beta)[u(y) - u(x)] \geq I(\alpha u(f) + (1-\alpha) u(x) e) + (1-\beta)[u(y) - u(x)] = I(\alpha u(f) + (1-\alpha) u(x) e) + k(1-\alpha)[u(y) - u(x)] \geq I(\alpha u(f) + (1-\alpha) u(y) e) = I(u(\alpha f + (1-\alpha) f_y))$. Hence, $\beta g + (1-\beta) f_y \succsim \alpha f + (1-\alpha) f_y$. Similarly, one can check the other part of A.4.2.

To see A.6, first fix $f_1, f_2 \in \mathcal{F}_0$ such that $(f_1, l) \succsim' (f_2, l)$ for all $l \in \mathcal{L}_1$. For all $t \in \mathbb{R}$, $v_{f_1}(t) \geq v_{f_2}(t)$, and thus $w(t, v_{f_1}(t)) \geq w(t, v_{f_2}(t))$. Hence, $E_{\bar{l}_{f_1}} u \geq E_{\bar{l}_{f_2}} u$. If $t \in [E_{\underline{l}_{f_2}} u, E_{\bar{l}_{f_2}} u] \setminus [E_{\underline{l}_{f_1}} u, E_{\bar{l}_{f_1}} u]$, then $t < E_{\underline{l}_{f_1}} u$, and thus $w(E_{\underline{l}_{f_1}} u, 1) > w(t, v_{f_1}(t)) \geq w(t, v_{f_2}(t))$. Hence, for all $t \in [E_{\underline{l}_{f_2}} u, E_{\bar{l}_{f_2}} u]$, there exists $t' \in [E_{\underline{l}_{f_1}} u, E_{\bar{l}_{f_1}} u]$ such that $w(t', v_{f_1}(t')) \geq w(t, v_{f_2}(t))$. Thus, $W(f_1) \geq W(f_2)$, i.e., $f_1 \succsim f_2$.

Next fix $N \in \mathbb{Z}_+$ and $f, f_1, ..., f_N \in \mathcal{F}_a$ such that $\max\{\underline{l}_{f_n} \mid n = 1, ..., N\} \succ \underline{l}_f$ and $\max\{(f_n, l) \mid i = 1, ..., N\} \succ' (f, l)$ for all l satisfying $\bar{l}_f \succsim l \succ \underline{l}_f$. Suppose that $W(f) = w(t, v_f(t))$, $t \in [E_{\underline{l}_f} u, E_{\bar{l}_f} u]$. If $t = E_{\underline{l}_f} u$, then $t < E_{\underline{l}_{f_n}} u$ for some $n \in \{1, ..., N\}$. Thus, $W(f) = t < E_{\underline{l}_{f_n}} u \leq W(f_n)$. If $t \in (E_{\underline{l}_f} u, E_{\bar{l}_f} u]$, then $v_f(t) < v_{f_n}(t)$ for some $n \in \{1, ..., N\}$. It is easy to see that there exists $t' \in (E_{\underline{l}_{f_n}} u, E_{\bar{l}_{f_n}} u)$ such that $(t', v_{f_n}(t')) \succ (t, v_f(t))$. Hence, $W(f_n) > W(f)$ as desired. □

References

[1] Anscombe, F. and Aumann, R. A definition of subjective probability, *The Annals of Mathematics and Statistics* **34** (1963), 199–205

[2] Eichberger, J., Grant, S., Kelsey, D. and Koshevoy, G. The α-MEU model: a comment, *Journal of Economic Theory* **146(4)** (2011), 1684–1698

[3] Ellsberg, D. Risk, ambiguity and the Savage axioms, *The Quarterly Journal of Economics* **75** (1961), 643–669

[4] Epstein, L. and Zhang, J. Subjective probabilities on subjectively unambiguous events, *Econometrica* **69** (2001), 265–306

[5] Ghirardato, P., Maccheroni, F. and Marinacci, M. Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory* **118** (2004), 133–173

[6] Gilboa, I. and Schmeidler D. Maximin expected utility with non-unique prior, *Journal of Mathematical Economics* **18** (1989), 141–153

[7] Grandmont, J. Continuity properties of a von Neumann-Morgenstern utility, *Journal of Economic Theory* **4** (1972), 45–57

[8] Hansen, L. and Sargent, T. Robust control and model uncertainty, *The American Economic Review* **91(2)** (2001), 60–66

[9] Herstein, I. N. and Milnor, J. An axiomatic approach to measurable utility, *Econometrica* **21** (1953), 291–29

[10] Klibanoff, P., Marinacci, M. and Mukerji, S. Definitions of ambiguous events and the smooth ambiguity model, *Economic Theory* **48(2)** (2011), 399–424

[11] Klibanoff, P., Marinacci, M. and Mukerji, S. A smooth model of decision making under ambiguity, *Econometrica* **73(6)** (2005), 1849–1892

[12] Knight, F. *Risk, Uncertainty and Profit*, Houghton Mifflin, Boston (1921)

[13] Maccheroni, F., Marinacci, M., and Rustichini, A. Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica* **74(6)** (2006), 1447–1498

[14] Nehring, K. Capacities and probabilistic beliefs: a precarious coexistence, *Mathematical Social Sciences* **38** (1999), 197–213

[15] Nehring, K. Is it possible to define subjective probabilities in purely behavioral terms? A comment on Epstein-Zhang (2001), Mimeo (2006)

[16] Savage, L. *Foundations of Statistics*, Wiley, New York (1954)

[17] Seo, K. Ambiguity and second-order belief, *Econometrica* **77(5)** (2009), 1575–1605

[18] Schmeidler, D. Subjective probability and expected utility without additivity, *Econometrica* **57** (1989), 571–587

[19] Strzalecki, T. Axiomatic foundations of multiplier preferences, *Econometrica* **79(1)** (2011), 47–73

[20] Zhang, J. Subjective Ambiguity, Expected Utility and Choquet Expected Utility, *Economic Theory* **20(1)** (2002), 159–181