Three representations of preferences with decreasing absolute uncertainty aversion

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Abstract

This paper axiomatizes a class of preferences displaying decreasing absolute uncertainty aversion, which allows a decision maker to be more willing to take uncertainty-bearing behavior when he becomes wealthier. In our first main result, we obtain three equivalent representations. The first is a variation on the constraint criterion of Hansen and Sargent (2001). The other two generalize Gilboa and Schmeidler (1989)'s maxmin criterion and Maccheroni, Marinacci and Rustichini (2006)'s variational representation.

This class, when restricted to preferences exhibiting constant absolute uncertainty aversion, is exactly Maccheroni, Marinacci and Rustichini (2006)'s variational preferences. In our second main result, we establish relationships among the representations for several important classes within variational preferences.

1 Introduction

It is a well-known economic phenomenon that people's risk aversion decreases with wealth. For example, consider a risk of winning or losing \$500 with equal probability. A person with initial wealth \$550 should be willing to pay more for insurance than should a person

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with wealth $\$50,000$. In this case, his preference is said to display *decreasing absolute risk aversion* (Arrow (1963), Pratt (1964)).

When applying this theory to macroeconomics and finance models, researchers typically assume either that the risk is objectively given, or that the decision maker has a subjective probability. However, empirical evidence shows that in real-life settings uncertainty is more relevant than risk.[1] That is, the probability distribution governing the potential outcomes is often unknown to the decision maker (Knight, 1921). Moreover, although the subjective probability assumption finds its axiomatic foundation in Savage (1954), Ellsberg (1961)'s famous thought experiments and many subsequent field experiments reveal that people's behavior violates Savage's axioms, suggesting that decision makers do not have a subjective probability.

Nevertheless, the wealth effect on uncertainty has not been sufficiently studied. Most commonly used models of preferences under uncertainty impose the restrictive assumption that the degree of a decision maker's uncertainty aversion is constant. However, it might be expected that wealthier people are more willing to take uncertainty-bearing behavior. If in the opening example the probability of winning and losing is unknown to the decision maker, he may still be willing to pay more for insurance when he has initial wealth \$550 than when he has initial wealth $\$50,000$. In this case, the decision maker's preference exhibits *decreasing absolute uncertainty aversion*.

This paper studies the effect of wealth on uncertainty aversion. Our first main result axiomatizes a class of preferences that display decreasing absolute uncertainty aversion. Three equivalent representations are obtained.[2]

All axioms considered in this paper are standard in the literature, with one important innovation — an axiom called *decreasing absolute uncertainty aversion* (hereafter DAUA).[3] Consider an act f as a state-contingent payoff profile, and call an act constant if it gives the same payoff in each state. The DAUA axiom requires that if an act f is weakly preferred to a constant act, then it is still weakly preferred after a common improvement in every

[1] For example, an investor chooses the optimal amount of investment in the presence of unknown random shocks that impinge on production, people decide how much to spend on health insurance without knowing exactly the chance of getting sick, and the environmental department makes regulations for a new technology based only on an estimation of the pollution rate.

[2] The results for preferences with increasing absolute uncertainty aversion are analogously obtained.

[3] Note that throughout the paper, "decreasing absolute uncertainty aversion" means "non-increasing absolute uncertainty aversion".

state for both acts. This implies that a decision maker becomes weakly more tolerant to the uncertainty of f as he gets wealthier. The DAUA axiom weakens Gilboa and Schmeidler (1989)'s certainty independence axiom and Maccheroni, Marinacci and Rustichini (2006)'s weak certainty independence axiom. Preferences satisfying our axioms are called *DAUA variational preferences*. DAUA variational preferences include several important classes of preferences which display constant absolute uncertainty aversion (see the second main result).

We obtain three different yet equivalent representations for DAUA variational preferences.[4] The first representation is a *variant constraint representation*:

$$V(f) = \min_{p \in \{p \in \Delta | d(p,B) \leq \eta(u(f))\}} E_p u(f). \tag{1}$$

It models situations in which the decision maker has a set B of best-guess priors but does not fully trust these priors. He considers all priors p within η distance of B, and evaluates f by its minimum expected utility over such neighborhood of B. The bound constraint η is a function of utility profiles. In particular, it weakly decreases in the ensured (or baseline) utility level of an act. This means that the decision maker becomes less concerned with robustness — equivalently, more tolerant to uncertainty — as he becomes better off overall.

The variant constraint representation in (1) is a variation on the *constraint criterion* introduced by Hansen and Sargent (2001) as a robust decision rule. A constraint criterion evaluates an act by

$$V(f) = \min_{p \in \{p \in \Delta | R(p\|q) \leq \eta\}} E_p u(f),$$

where q is a best-guess prior and $R(p\|q)$ is the relative entropy of p with respect to q. An important difference when compared to (1) is that here, η is constant over all the utility profiles, which implies that the degree of the decision maker's uncertainty aversion is *fixed*. While an axiomatization of the constraint criterion is still an open question, our result provides an axiomatic foundation for a variant constraint criterion which is of the same spirit and allows decreasing absolute uncertainty aversion.

The second representation is a *weighted maxmin representation*:

$$V(f) = \lambda(u(f)) \min_{p \in C} E_p u(f) + (1 - \lambda(u(f))) \max_{p \in C} E_p u(f). \tag{2}$$

[4] A representaion for a preference $\succsim$ over acts is a function V of acts such that $f \succsim g \Leftrightarrow V(f) \geq V(g)$ for all acts f, g.

The decision maker considers C as the set of all possible priors and evaluates an act by a weighted average of the minimum and maximum expected utility over C. The weight λ on the worst case weakly decreases in the baseline utility, meaning that the decision maker becomes more optimistic (less uncertainty averse) with the increase of his ensured payoffs.

The third representation is the *DAUA variational representation*:

$$V(f) = \min_{p \in \Delta}[E_p u(f) + c(E_p u(f), p)], \tag{3}$$

where c is a cost function of expected utilities and priors. The cost function basically plays the role of restricting the priors under consideration. In particular, c weakly increases in the utility term. It turns out that when the baseline utility of an act rises, it will be evaluated by a more "favorable" prior. When c is constant in utility, (3) is reduced to Maccheroni, Marinacci and Rustichini (2006)'s variational representation, which represents the class of variational preferences that display constant absolute uncertainty aversion.

The three representations relate to each other in a nice way. The set of the priors with zero cost at any utility level is exactly the set B, while the set C contains precisely the priors that have finite cost at some utility level. The set B is a subset of C. They provide, respectively, an upper and lower bound for the evaluation. The value of each act is below the worst expected utility over B and above that over C. Moreover, the bounds are tight. At best, the decision maker considers only the priors in B; at worst, he considers all the priors in C. Gilboa and Schmeidler (1989)'s maxmin preferences are exactly characterized by the condition $B = C$.

Our second main result is to establish relationships among the representations for several important classes of preferences in the literature, when we restrict our attention to preferences displaying constant absolute uncertainty aversion. One finding is that the representations for three nested classes of preferences — variational, maxmin and constraint preferences — in fact "commute" with other other. More precisely, while Maccheroni, Marinacci and Rustichini (2006) build the connection between maxmin preferences (and thus its subclass, constraint preferences) and multiplier preferences by showing that both belong to a larger class of variational preferences, our result suggests that the converse is true as well. That is, variational preferences also live in a class of generalized maxmin preferences and in a class of a constraint type of preferences (see Figure 1).

This result comes from the following observation. The subclass of DAUA variational preferences with constant absolute uncertainty aversion is exactly the class of variational preferences. The three representations above, when restricted to variational preferences, also

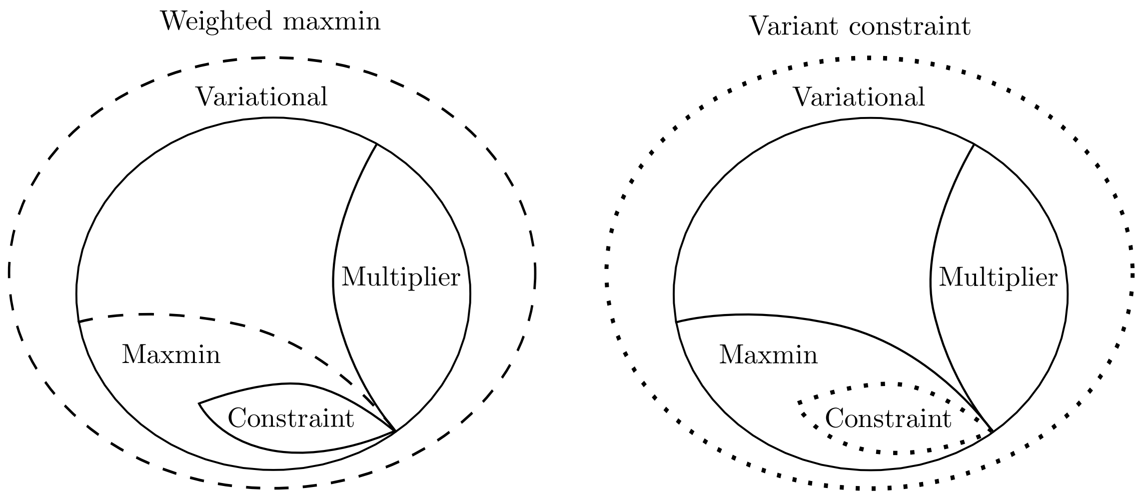

Figure 1: Relations of preferences that display constant absolute uncertainty aversion.

give three equivalent representations. In the first two representations, the distance constraint η and the weight function λ become constant in the baseline utility of an act. Representation (3), with the cost function c no longer depending on the utility term, is reduced to the variational representation obtained by Maccheroni, Marinacci and Rustichini (2006).

The other finding is an equivalent representation for the multiplier criterion[5] introduced by Hansen and Sargent (2001) as the second robust decision rule. This representation closely resembles the constraint criterion and clearly shows the relationships between the two criteria. Hansen and Sargent (2001) establish the "equivalence" between constraint and multiplier criteria in a dynamic resource allocation problem by showing that both rules imply the same optimal solution. However, they generally give different rankings of acts other than the optimal one. we further clarify their relationship by an equivalent representation for multiplier criterion:

$$V(f) = \min_{p\in\{p\in\Delta | R(p\|q)\leq\eta(u(f))\}} E_p u(f),$$

where η is a function of utility profiles corresponding to the parameter in the multiplier criterion. This shows that the difference between the constraint and multiplier criteria lies in the distance constraint η. For the constraint criterion, η is a constant function. For the multiplier criterion, η is a particular function which is constant only if it is constantly 0.

This paper is organized as follows. Section 2 states the axioms. Section 3 presents the

[5]The multiplier criterion evaultes an act by $V(f) = \min_{p\in\Delta}[E_p u(f) + \theta R(p\|q)]$, where $\theta \in (0, \infty]$ is a parameter.

three representations. Section 4 studies several subclasses of DAUA variational preferences and provides another representation for multiplier preferences. Section 5 concludes.

1.1 Related literature

Although decreasing absolute uncertainty aversion is a natural analogy of the classic concept of decreasing absolute risk aversion, there are only two recent papers addressing this issue. Klibanoff, Marinacci and Mukerji (2005) characterize a class of preferences such that a decision maker has a von Neumann-Morgenstern utility over the outcomes and treats each act as a prior-contingent expected utility function. The decision maker has a second-order belief over all priors and evaluates an act by the expectation of an increasing transformation of its prior-dependent expected utility function. The expectation is taken with respect to his second-order belief over priors, and the increasing transformation is viewed as a second-order utility function. They characterize preferences with decreasing absolute uncertainty aversion by the properties of a second-order utility function, analogous to the approach used with decreasing absolute risk aversion.

Chambers, Grant, Polak and Quiggin (2012) study a two-parameter model where a decision maker has a baseline prior and a measure of dispersion of acts. The decision maker evaluates an act based on its mean with respect to the baseline prior and its dispersion. They represent preferences with decreasing absolute uncertainty aversion by the property of an aggregating function of mean and dispersion. Their DAUA axiom is stronger than the one in this paper. Their axiom compares the effects of improving a certainty part on all pairs of acts where one act is more dispersed than the other.

Our paper adopts a different approach and studies a different model of preferences with decreasing absolute uncertainty aversion. This approach accommodates situations where the decision maker does not have a second-order belief or a baseline prior, but only a range of estimated priors.

2 Setup

We denote by $\mathbb{R}$ the set of all the reals, and $\mathbb{R}_+$ the set of all the non-negative reals. Let S be a set of *states of the world*. A subset of S is called an *event*. We assume that S is finite and has cardinality n. The set of all probabilities on S is denoted by Δ. We identify Δ with the unit simplex in $\mathbb{R}^n$, i.e., the set $\{(p_1, ..., p_n) \in \mathbb{R}^n | \sum_{i=1}^{n} p_i = 1 \text{ and } p_i \geq 0 \text{ for all } i\}$, and Δ is regarded as a metric space with the Euclidean metric.

Let X be a set of *outcomes*. We follow Maccheroni, Marinacci and Rustichini (2006) to assume that X is a convex subset of some vector space. Note that it includes Anscombe and Aumann (1963)'s classic setting where X is the set of all lotteries on a set of prizes. An *act* is a function $f : S \to X$. Let $\mathcal{F} = X^S$ be the set of all acts. Given an outcome $x \in X$, with a slight abuse of notation, we also denote by x the *constant act* which assigns x to all $s \in S$, and identify X with the set of all constant acts. Given $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we define the convex combination $\alpha f + (1 - \alpha)g$ as an act in $\mathcal{F}$ such that $[\alpha f + (1 - \alpha)g](s) = \alpha f(s) + (1 - \alpha)g(s)$ for all $s \in S$.

A decision maker's *preference* is a binary relation $\succsim$ on $\mathcal{F}$. Let $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$ as usual. Given $f \in \mathcal{F}$, an element $x_f \in X$ is a *certainty equivalent* of f if $x_f \sim f$. A function $V : \mathcal{F} \to \mathbb{R}$ *represents* $\succsim$ on $\mathcal{F}$ if $f \succsim g \Leftrightarrow V(f) \geq V(g)$ for all $f, g \in \mathcal{F}$.

3 Axioms

Consider the following axioms for $\succsim$.

A.1. Weak Order. (1) For all $f, g \in \mathcal{F}$, either $f \succsim g$ or $g \succsim f$.

(2) For all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

A.2. Decreasing Absolute Uncertainty Aversion. For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in (0, 1)$, if either f is a constant act or $y \succsim x$, then

$$\alpha f + (1 - \alpha)x \succsim \alpha z + (1 - \alpha)x$$
$$\Rightarrow \alpha f + (1 - \alpha)y \succsim \alpha z + (1 - \alpha)y.$$

A.3. Continuity. For all $f, g, h \in \mathcal{F}$, the set $\{\alpha \in [0, 1] | \alpha f + (1 - \alpha)g \succsim h\}$ and the set $\{\alpha \in [0, 1] | h \succsim \alpha f + (1 - \alpha)g\}$ are closed in $\mathbb{R}$.

A.4. Monotonicity. For all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for every $s \in S$, then $f \succsim g$.

A.5. Uncertainty Aversion. For all $f, g \in \mathcal{F}$ and $\alpha \in (0, 1)$, if $f \sim g$, then $\alpha f + (1 - \alpha)g \succsim f$.

A.6. Unboundedness. There exist $x, y \in X$ such that (1) $x \succ y$, and (2) for each $\alpha \in (0, 1)$, there are $z, z' \in X$ satisfying $\alpha z + (1-\alpha)y \succsim x \succ y \succsim \alpha z' + (1-\alpha)x$.

A preference $\succsim$ on $\mathcal{F}$ is called a *DAUA variational preference* if it satisfies Axiom A.1 - A.6.

Axiom A.1, A.3, A.4 and A.5 are standard in the literature (see e.g. Anscombe and Aumann (1963), Schmeidler (1989) and Gilboa and Schmeidler (1989)). Weak order requires preferences to be complete and transitive. Continuity states that preferences are continuous with respect to the coefficients of convex combination of acts. Monotonicity assumes that the decision maker ranks the outcomes as constant acts, and that an act is weakly preferred if it assigns a weakly better outcome in each state. Uncertainty aversion captures the decision maker's preference for hedging under uncertainty.

Axiom A.6 is stronger than the usual non-degeneracy axiom. The non-degeneracy axiom asks that there exists at least one act which is strictly preferred to some other. Axiom A.6 enforces the obtained utility function on X representing $\succsim$ on constant acts to range over all the reals. This axiom is commonly used in the recent literature (see e.g. Kopylov (2001), Maccheroni, Marinacci and Rustichini (2006), Strzalecki (2011b) and Grant and Polak (2011)). In some places it is a technical assumption which simplifies the analysis, while in the other places it is indispensable for some desirable results. In this paper, A.6 is necessary since our representation crucially relies on the preference for the "limiting acts", i.e., the acts causing extremely good or bad outcomes in all the states.

The rest of this section is devoted to A.2. Maccheroni, Marinacci and Rustichini (2006) introduce the weak certainty independence axiom which weakens the certainty independence axiom of Gilboa and Schmeidler (1989). Our A.2 is a further weakening of the weak certainty independence.

A.2.1. Certainty Independence. For all $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0, 1)$,

$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x.$$

A.2.2. Weak Certainty Independence. For all $f, g \in \mathcal{F}$, $x, y \in X$ and $\alpha \in (0, 1)$,

$$\begin{aligned} &\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \\ &\qquad \Rightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y. \end{aligned} \tag{4}$$

Maccheroni, Marinacci and Rustichini (2006) show that a preference $\succsim$ satisfies A.2.1 if and only if for all $f, g \in \mathcal{F}$, $x, y \in X$ and $\alpha, \beta \in (0, 1]$,

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x$$
$$\Rightarrow \beta f + (1-\beta)y \succsim \beta g + (1-\beta)y.$$

Thus A.2.2 weakens A.2.1 to require that the preference of two acts is only independent of the constant acts that they are mixed with, but not the weights in the mixing. Grant and Polak (2011) show that under the previous axioms, A.2.2 is equivalent to their constant absolute uncertainty aversion axiom which assumes the same condition (4) to hold only when g is constant.[6]

A.2.3. Constant Absolute Uncertainty Aversion. For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)x \succsim \alpha z + (1-\alpha)x$$
$$\Rightarrow \alpha f + (1-\alpha)y \succsim \alpha z + (1-\alpha)y. \tag{5}$$

Our Axiom A.2 naturally extends A.2.3 to decreasing absolute uncertainty aversion by assuming (5) to hold either when f is constant or $y \succsim x$. In this way, it differentiates the effect of changing a certainty part on constant acts and that on non-constant acts.

First suppose that f is constant. Then A.2 is essentially von Neumann-Morgenstern's independence axiom on constant acts. It is the key to get an affine utility function, say u, to represent a preference on constant acts. Hence, changing a certainty part in *any* constant act, say from x to y by $1-\alpha$ proportion, the change in its utility is $(1-\alpha)(u(y) - u(x))$. In this sense, changing a certainty part generates the *same* effect on all constant acts. Thus, the preference is preserved under such a change.

Second, if f is not constant, then the preference is preserved only when $y \succsim x$. While the effect of increasing a certainty part on constant acts can be normalized by the analysis above, (4) means that increasing a certainty part creates weakly larger improvement on a non-constant act than on a constant act. Equivalently, if the uncertainty of a non-constant act is tolerable when compared to a constant act, then it is even more tolerable as the certainty part grows.

The following example is a variation on Ellsberg (1961)'s thought experiment. It shows different behavioral implications of A.2 and the three axioms above.

[6] Actually Grant and Polak (2011) show the equivalence under A.1, A.3, a weaker version of A.4 and A.6

Example 1. An urn contains 100 balls, of which 33 are red, and 67 are either black or yellow. A ball is drawn from the urn. For each $t \geq 0$, r_t denotes the act "betting on red". It pays $100 + t$ dollars if the ball is red and t dollars otherwise. Let b_t denote the act "betting on black", and its payoff is analogously given. See the table below.

Table 1: Payoffs of r_t and b_t

$t \geq 0$	Red	Black	Yellow
r_t	100+t	t	t
b_t	t	100+t	t

For instance, $t = 0$ and $t = 10^4$.

Table 2: Payoffs of r_0 and b_0

$t = 0$	Red	Black	Yellow
r_0	100	0	0
b_0	0	100	0

Table 3: Payoffs of r_{10^4} and b_{10^4}

$t = 10^4$	Red	Black	Yellow
r_{10^4}	10,100	10,000	10,000
b_{10^4}	10,000	10,100	10,000

Suppose that the decision maker's preference $\succsim$ satisfies A.1, A.3, A.4 and A.6, and assume for simplicity that he is risk neutral. Then A.2.1, A.2.2 and A.2.3 each implies that

$$\text{either } r_t \succsim b_t \text{ for all } t, \text{ or } b_t \succsim r_t \text{ for all t.}$$

However, our A.2 allows the existence of a threshold $\bar{t}$ such that

$$r_t \succsim b_t \text{ for all } t \leq \bar{t}, \text{ and } b_t \succsim r_t \text{ for all } t \geq \bar{t}.$$

Hence, A.2 accommodates the phenomenon that people become more willing to take uncertainty-bearing behavior as the baseline wealth increases.

4 Representations

Let $u : X \to \mathbb{R}$ be a utility function of outcomes. Given $f \in \mathcal{F}$, let $u(f)$ denote a function in $\mathbb{R}^S$ assigning $u(f(s))$ to each $s \in S$. Thus $u(f)$ transfers each act f to a state-contingent utility function. Let $I : u(X)^S \to \mathbb{R}$ be a functional on all state-contingent utility functions. We say that I is weakly increasing if $I(\varphi) \geq I(\psi)$ whenever $\varphi, \psi \in u(X)^S$ and $\varphi(s) \geq \psi(s)$ for all $s \in S$. Let $\mathbb{1}$ denote a function in $\mathbb{R}^S$ such that $\mathbb{1}(s) = 1$ for all $s \in S$. Similarly, for any $t \in \mathbb{R}$, $t\mathbb{1}$ denotes a function in $\mathbb{R}^S$ that gives t to each state $s \in S$. Define $\mathbb{R}\mathbb{1} = \{t\mathbb{1} \in \mathbb{R}^S | t \in \mathbb{R}\}$ to be the set of all constant utility functions on S.

Given $\varphi \in \mathbb{R}^S$, $p \in \Delta$, we denote by $E_p\varphi$ the expected value of φ with respect to p. Let $d(p, q)$ denote the Euclidean distance between two priors $p, q \in \Delta$, $d(p, A)$ that between a prior $p \in \Delta$ and a set $A \subseteq \Delta$, and $d(A, B)$ that between two sets $A, B \subseteq \Delta$. Lastly, we equip the space $\mathbb{R}^S$ with the topology induced by the supremum norm.

4.1 Variant constraint representation

Definition 1. *A variant constraint representation of a preference* $\succsim$ *is a triple* $\langle u, B, \eta \rangle$ *such that*
(1) $u : X \to \mathbb{R}$ *is a non-constant affine utility function, B is a non-empty closed convex subset of* Δ*, and* $\eta : u(X)^S \to \mathbb{R}_+$ *is a distance constraint function;*
(2) for $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{p \in \{p \in \Delta | d(p,B) \leq \eta(\varphi)\}} E_p\varphi, \ \forall \varphi \in u(X)^S, \tag{6}$$

I is weakly increasing and quasi-concave;
(3) for $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)), \ \forall f \in \mathcal{F}, \tag{7}$$

V represents $\succsim$*.*

The interpretation is that the decision maker considers each act as a state-contingent utility function. He has a set B of approximating or best-guess priors, but he does not fully trust them. To make decisions robust to prior misestimation, he evaluates an act by the minimum expected utility over all the priors within η distance of the approximating ones. The distance constraint η measures the degree of his concern for prior misestimation, and it is a function of utility profiles.

Theorem 1. *The following statements are equivalent.*

(1) A preference $\succsim$ *satisfies A.1 - A.6.*

(2) There exist an affine onto function $u : X \to \mathbb{R}$*, a non-empty closed convex set* $B \subseteq \Delta$ *and a function* $\eta : \mathbb{R}^S \to \mathbb{R}_+$ *such that*

(i) $\langle u, B, \eta \rangle$ *is a variant constraint representation of* $\succsim$*;*

(ii) η *is continuous on* $\mathbb{R}^S \setminus \mathbb{R}\mathbb{1}$*,* $\eta(\varphi + t\mathbb{1})$ *weakly decreases in* t *for all* $\varphi \in \mathbb{R}^S$*, and* $\lim_{k \searrow 0} \lim_{t \to \infty} \eta(k\varphi + t\mathbb{1}) = 0$.

Moreover, if $\langle u', B', \eta' \rangle$ *also satisfies the conditions in (2), then* $u' = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*,* $B' = B$*, and* $\eta'(a\varphi + b) = \eta(\varphi)$ *for all* $\varphi \in \mathbb{R}^S$ *such that* $I(\varphi) \neq \min_{s \in S} \varphi(s)$ *where* I *is given by* $\langle u, B, \eta \rangle$ *as in* (6).

The distance constraint function η has three properties. First, η is continuous at every non-constant $\varphi \in \mathbb{R}^S$. When φ is constant, the value of $I(\varphi)$ is identical for any distance constraint. However, I is still continuous on the whole domain $\mathbb{R}^S$ of state-contingent utility functions.

The second property of η is exactly the result of A.2. As the certainty utility increases, the decision maker may become more tolerant to prior misestimation, and thus reduces the range of priors under consideration. This implies that for I in (6), $I(\varphi + t\mathbb{1}) \geq I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \geq 0$. Clearly, the equality holds when φ is constant. This shows that increasing the baseline utility generates weakly more improvement on a non-constant act than on a constant one.

The third property of η reveals the decision maker's uncertainty aversion in the limiting case. It says that the decision maker tends not to consider any prior outside the best-guess set in the "extremely good" situation where first the baseline utility increases to ∞, and second the scale of the uncertain part diminishes to 0. To understand the latter, see the following example. It is a variation on Maccheroni, Marinacci and Rustichini (2006)'s Example 2.

Example 2. An urn contains 100 balls, of which 33 are red, and 67 are either black or yellow. A ball is drawn from the urn. For each $t > 0$, the act r_t, betting on red, pays t dollars if the ball is red, and t cents otherwise. The act b_t, betting on black, is defined analogously. See the following table of payoffs.

Table 4: Payoffs of r_t and b_t

$t > 0$	Red	Black	Yellow
r_t	t	0.01t	0.01t
b_t	0.01t	t	0.01t

For example, $t = 10$ and $t = 10^4$.

Table 5: Payoffs of r_{10} and b_{10}

$t = 10$	Red	Black	Yellow
r_{10}	10	0.1	0.1
b_{10}	0.1	10	0.1

Table 6: Payoffs of r_{10^4} and b_{10^4}

$t = 10^4$	Red	Black	Yellow
r_{10^4}	10,000	100	100
b_{10^4}	100	10,000	100

The scale of money payment is measured by t. Assume for simplicity that the decision maker displays constant relative risk aversion $\rho \in (0, 1)$. Then t is basically the same as the utility scalar k in Theorem 1 (2/ii). The decision maker may become more willing to take the uncertainty-bearing behavior when the payoff scale t decreases. As a result, there may exist a threshold value $\bar{t}$ such that

$$b_t \succsim r_t \text{ for all } t \leq \bar{t}, \text{ and } r_t \succsim b_t \text{ for all } t \geq \bar{t}.$$

This can be the case when his preference satisfies A.1, A.3 - A.6, and A.2.2 — the weak certainty independence[7]. In this case, $I(k\varphi) \leq kI(\varphi)$ for all $\varphi \in \mathbb{R}^S$ and $k \geq 1$. Alternatively, in our representation, it means that $\eta(\varphi) \leq \eta(k\varphi)$ for all $\varphi \in \mathbb{R}^S$ such that $I(\varphi) \neq \min_S \varphi(s)$ and $k \geq 1$. If $I(\varphi) = \min_S \varphi(s)$, then $\eta(\varphi)$ and $\eta(k\varphi)$ can take any value which is above some lower bound. This property is referred as increasing relative uncertainty aversion (see e.g. Strzalecki (in press), Chateauneuf and Faro (2009); see also Section 5.2 for more discussion). It is basically the result of A.2.2 and A.5. If A.2.2 is weakened to A.2, then this property holds only when the certainty utility decreases to $-\infty$. More precisely, $\lim_{t\to\infty} \eta(\varphi - t\mathbb{1}) \leq \lim_{t\to\infty} \eta(k\varphi - t\mathbb{1})$ for all $\varphi \in \mathbb{R}^S$ such that $I(\varphi) \neq \min_S \varphi(s)$ and $k \geq 1$. Intuitively, it says that the payoff scale becomes an issue when the baseline utility is sufficiently low.

4.2 Weighted maxmin representation

Definition 2. *A* weighted maxmin representation *of a preference* $\succsim$ *is a triple* $\langle u, C, \lambda \rangle$ *such that*
(1) $u : X \to \mathbb{R}$ *is a non-constant affine utility function,* C *is a non-empty closed convex subset of* Δ*, and* $\lambda : u(X)^S \to [0, 1]$ *is a weight function;*
(2) for $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \lambda(\varphi) \min_{p\in C} E_p\varphi + (1 - \lambda(\varphi)) \max_{p\in C} E_p\varphi \ \forall\varphi \in u(X)^S, \tag{8}$$

I is weakly increasing and quasi-concave;
(3) for $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)) \ \forall f \in \mathcal{F},$$

V *represents* $\succsim$.

The interpretation is that the decision maker considers each act f as a utility profile $u(f)$. He believes that C is the set of all possible priors, and evaluates an act by a weighted average of the best and worst expected utility over the priors in C. The weight λ that he puts on the worst case is a function of utility profiles.

Theorem 2. *The following statements are equivalent.*

(1) A preference $\succsim$ *satisfies A.1 - A.6.*

[7]See Maccheroni, Marinacci and Rustichini (2006).

(2) There exist an affine onto function $u : X \to \mathbb{R}$, a non-empty closed convex set $C \subseteq \Delta$ and a function $\lambda : \mathbb{R}^S \to [0, 1]$ such that
(i) $\langle u, C, \lambda \rangle$ is a weighted maxmin representation of $\succsim$;
(ii) λ is continuous on $\{\varphi \in \mathbb{R}^S \,|\, \min_{p\in C} E_p\varphi < \max_{p\in C} E_p\varphi\}$, $\lambda(\varphi + t\mathbb{1})$ weakly decreases in t for all $\varphi \in \mathbb{R}^S$, and $\lim_{k\to\infty} \lim_{t\to\infty} \lambda(k\varphi - t\mathbb{1}) = 1$.

Moreover, if $\langle u', C', \lambda' \rangle$ also satisfies the conditions in (2), then $u' = au+b$ for some $a > 0$ and $b \in \mathbb{R}$, $C' = C$, and $\lambda'(a\varphi + b) = \lambda(\varphi)$ for all $\varphi \in \mathbb{R}^S$ such that $\min_{p\in C} E_p\varphi < \max_{p\in C} E_p\varphi$.

The properties of the weight function λ are analogous to those of η. The limiting condition of λ says that the decision maker exhibits significant uncertainty aversion in the "extremely bad" situation, so that he tends to consider only the worst case. The "extremely bad" situation means that first, the baseline utility drops to $-\infty$, and second, the scale of the uncertain part expands to ∞.

Next we give a short discussion about the relationship between the variant constraint representation in Theorem 1 and the weighted maxmin representation in Theorem 2. In fact, they are dual to each other in the sense that they respectively describe a decision maker's behavior in the "extremely good" situation and in the "extremely bad" situation. The set B in Theorem 1 reveals the priors that the decision maker always consider in minimizing expected utilities. He would never be "bold" enough to ignore any of them in any situation. The set C in Theorem 2 reveals the priors that the decision maker would minimize over when evaluating some act.

Corollary 1. *Let $\langle u, B, \eta \rangle$ be the variant constraint representation as in Theorem 1 and $\langle u, C, \lambda \rangle$ the weighted maxmin representation as in Theorem 2. Then $B \subseteq C$.*

These two sets respectively provide an upper and lower bound for the evaluation. The value of any act is below the worst expected utility over B and above that over C. The two bounds are tight. At best, the decision maker only considers the priors in B; at worst, he considers all the priors in C.

4.3 DAUA variational representation

Definition 3. *A DAUA variational representation of a preference $\succsim$ is a pair $< u, c >$ such that*
(1) $u : X \to \mathbb{R}$ is a non-constant affine utility function, and $c : u(X) \times \Delta \to [0, \infty]$ is a lower

semicontinuous cost function satisfying that (i) the function $c(t, p) + t : u(X) \times \Delta \to (-\infty, \infty]$ *is quasi-convex, (ii)* $c(t, p)$ *is weakly increasing in* t *for each* $p \in \Delta$*, and (iii)* $\inf_{p\in\Delta} c(t, p) = 0$ *for each* $t \in u(X)$*;*
(2) for $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{p\in\Delta}[E_p\varphi + c(E_p\varphi, p)],\ \forall\varphi \in u(X)^S, \tag{9}$$

I is continuous;
(3) for $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)),\ \forall f \in \mathcal{F},$$

V represents $\succsim$*.*

The DAUA variational representation generalizes Maccheroni, Marinacci and Rustichini (2006)'s variational representation.

Definition 4. *A variational representation of a preference* $\succsim$ *is a pair* $\langle u, c\rangle$ *such that*
(1) $u : X \to \mathbb{R}$ *is a non-constant affine utility function, and* $c : \Delta \to [0, \infty]$ *is a lower semicontinuous convex cost function satisfying* $\inf_{p\in\Delta} c(p) = 0$*;*
(2) for $I : u(X)^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{\Delta}[E_p\varphi + c(p)],\ \forall\varphi \in u(X)^S,$$

and for $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)),\ \forall f \in \mathcal{F},$$

V represents $\succsim$*.*

A preference $\succsim$ *admitting a variational representation is called a variational preference.*

The DAUA variational representation generalizes the cost function in the variational representation by allowing a prior to have different costs at different utility levels. In particular, the cost of a prior weakly increases in utility, which corresponds to the DAUA axiom. Indeed, when the certainty utility of an act increases, say by t, the expected utility with respect to every prior is increased by t. Moreover, the cost of each prior is measured at a higher utility level and thus also weakly increases. As a result, the value of the act is increased by at least t, or equivalently, the act is evaluated by a more "favorable" prior. This means that decision maker becomes less averse to uncertainty when he becomes better off overall. With the cost function being constant in the utility term, the variational representation implies constant absolute uncertainty aversion.

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) obtain a general representation for a class of uncertainty averse preferences. The DAUA variational preferences include variational preferences, and are contained in this class of uncertainty averse preferences. By restricting the result of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) to the preferences displaying decreasing absolute uncertainty aversion, we obtain the following representation.

Proposition 1. *The following statements are equivalent.*

(1) A preference $\succsim$ *satisfies A.1 - A.6.*

(2) There exist an affine onto function $u : X \to \mathbb{R}$ *and a function* $c : \mathbb{R} \times \Delta \to [0, \infty]$ *such that* $\langle u, c \rangle$ *is a DAUA variational representation of* $\succsim$.

Moreover, u *is unique up to a positive affine transformation. For each* u*,* c *is uniquely given by* $c(t, p) = \sup\{u(x_f) - t | E_p u(f) \leq t, f \in \mathcal{F}\}$ *where* $(t, p) \in \mathbb{R} \times \Delta$.

Lastly, let B *and* C *be given as in Theorem 1 and Theorem 2 respectively. Then* $B = \{p \in \Delta | c(t, p) = 0$ *for all* $t \in \mathbb{R}\}$*, and* $C = \{p \in \Delta | c(t, p) < \infty\}$ *for any* $t \in \mathbb{R}$.

This representation is related to the previous two via the cost function. The set of priors with zero cost at any utility level coincides with the set B of all best-guess priors. Both sets reveal the priors which are always considered in the evaluation.

Besides, for any utility level $t \in \mathbb{R}$, the set of priors that have finite cost at t coincides with C. This means two things. First, if a prior has a finite cost at some utility level, then it has a finite cost at all utility levels. This property is not obvious. It basically comes from the interaction of A.2, A.5 and A.6. Second, the set of all priors that have finite cost at some utility level coincides with C. They reveal the priors that the decision maker would consider in some situation.

It follows that if a prior has an infinite cost at some utility level, then it has an infinite cost at all utility levels. The set of priors that have infinite cost at all utility levels is exactly $\Delta \setminus C$. Those priors are always excluded from consideration. No act is evaluated below the expected utility over C.

5 Special cases

5.1 Variational preferences

The class of variational preferences introduced by Maccheroni, Marrinacci and Rustichini (2006) is a subclass of DAUA variational preferences. They satisfy the stronger A.2.2 and thus display constant absolute uncertainty aversion. When restricting the previous results to this subclass, we again get three equivalent representations. The last representation is reduced to the variational representation. Hence, we find two other equivalent representations for variational preferences.

Proposition 2. *Suppose that* $\succsim$ *is a DAUA variational preference. Let* $\langle u, B, \eta \rangle$ *be the variant constraint representation as in Theorem 1,* $\langle u, C, \lambda \rangle$ *the weighted maxmin representation as in Theorem 2, and* $\langle u, c \rangle$ *the DAUA variational representation as in Proposition 1. Then the following statements are equivalent.*

(1) The preference $\succsim$ *satisfies A.2.2.*

(2) For the distance constraint function η*,* $\eta(\varphi + t\mathbb{1}) = \eta(\varphi)$ *for all* $t \in \mathbb{R}$ *and* $\varphi \in \mathbb{R}^S$*.*

(3) For the weight function λ*,* $\lambda(\varphi + t\mathbb{1}) = \lambda(\varphi)$ *for all* $t \in \mathbb{R}$ *and* $\varphi \in \mathbb{R}^S$*.*

(4) For the cost function c*,* $c(t, p) = c(t', p)$ *for all* $t, t' \in \mathbb{R}$ *and* $p \in \Delta$*.*

When the three representations are restricted to variational preferences, the distance constraint function η and the weight function λ do not change with the baseline utility, and the cost function c does not depend on the utility term. This implies for the functional form I in either (6), (8) or (9) that $I(\varphi + t\mathbb{1}) = I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \in \mathbb{R}$. It means that adding t units of utility in each state for any utility profile φ, the change of the value is always t, which is independent of φ. Thus, changing the certainty part generates the same effect on all acts. Hence, the degree of the decision maker's uncertainty aversion is fixed.

Proposition 2 shows that the variant constraint representation satisfying condition (2) and the weighted maxmin representation satisfying condition (3) are equivalent to the variational representation. The two equivalent representations provide different perspectives on variational preferences which are not obvious from the variational representation.

Corollary 2. *Suppose that* $\succsim$ *is a variational preference. Let* $\langle u, B, \eta \rangle$*,* $\langle u, C, \lambda \rangle$ *and* $\langle u, c \rangle$ *be the three equivalent representations for* $\succsim$ *as in Proposition 2. Then the following statements hold.*

(1) For the distance constraint function η, $\eta(k\varphi) \geq \eta(\varphi)$ for all $k \geq 1$ and $\varphi \in \mathbb{R}^S$ such that $I(\varphi) \neq \min_{s\in S} \varphi(s)$ where I is given as in (6), and $\lim_{k\searrow 0} \eta(k\varphi) = 0$ for all $\varphi \in \mathbb{R}^S$.

(2) For the weight function λ, $\lambda(k\varphi) \geq \lambda(\varphi)$ for all $k \geq 1$ and $\varphi \in \mathbb{R}^S$, and $\lim_{k\to\infty} \lambda(k\varphi) = 1$ for all $\varphi \in \mathbb{R}^S$.

(3) The set $B = \{p \in \Delta | c(p) = 0\}$ and the set $C = \{p \in \Delta | c(p) < \infty\}$.

The first two representations directly show that variational preferences exhibit relative increasing uncertainty aversion. This property has been discussed in Section 4.1. It mainly comes from the interaction of A.2.2 and A.5. As a result, when the size of φ is scaled up, the decision maker becomes weakly more averse to uncertainty. Thus he considers more priors which are farther away from the best-guess ones, or he puts more weight on the worst case. Moreover, as the wealth effect is assumed away, only the scale effect plays a role in the limiting conditions. When the scalar k of a utility profile diminishes to 0, then the decision maker tends to evaluate an act by the worst expected utility over only the priors in B. When k expands to ∞, he tends to consider all the priors in C.

Lastly, the set of zero-cost priors and the set of finite-cost priors in the variational representation coincide respectively with B and C. This provides a further understanding of the cost function c.

5.2 Maxmin preferences

Definition 5. *A maxmin representation of a preference $\succsim$ is a pair $\langle u, C\rangle$ such that*
(1) $u : X \to \mathbb{R}$ is a non-constant affine utility function, and C is a non-empty closed convex subset of Δ;
(2) for $V : \mathcal{F} \to \mathbb{R}$ defined as

$$V(f) = \min_{p\in C} E_p u(f), \ \forall f \in \mathcal{F},$$

V represents $\succsim$.

A preference $\succsim$ admitting a maxmin representation is called a maxmin preference.

Gilboa and Schmeidler (1989)'s maxmin preferences are special cases of variational preferences. Maxmin preferences satisfy the stronger A.2.1. They display not only constant absolute uncertainty aversion, but also constant relative uncertainty aversion.

The maxmin representation can be viewed as a variational representation with the cost function satisfying that $c(p) = 0$ for all $p \in C$ and $c(p) = \infty$ otherwise. The following result shows how the maxmin model fits in our representations.

Proposition 3. *Suppose that $\succsim$ is a DAUA variational preference. Let $\langle u, B, \eta \rangle$ be the variant constraint representation as in Theorem 1, $\langle u, C, \lambda \rangle$ the weighted maxmin representation as in Theorem 2, and $\langle u, c \rangle$ the DAUA variational representation as in Proposition 1. Then the following statements are equivalent.*

(1) The preference $\succsim$ satisfies A.2.1.

(2) For the distance constraint function η, $\eta(\varphi) = 0$ for all $\varphi \in \mathbb{R}^S$ such that $I(\varphi) \neq \min_{s \in S} \varphi(s)$ where I is given as in (6).

(3) For the weight function λ, $\lambda(\varphi) = 1$ for all $\varphi \in \mathbb{R}^S$.

(4) The set B coincides with the set C.

(5) For the cost function c, $c(t, p) = 0$ for all $t \in \mathbb{R}$ and $p \in B$, and $c(t, p) = \infty$ for all $t \in \mathbb{R}$ and $p \in \Delta \setminus B$.

If a decision maker has a maxmin preference, then the set of best-guess priors is the set of all possible priors. He always evaluates an act by the worst expected utility over this set. Moreover, at any utility level, the priors with finite cost are exactly those with zero cost. The equivalence between (1) and (5) is essentially Proposition 19 of Maccheroni, Marinacci and Rustchini (2006).

5.3 Constraint preferences and multiplier preferences

Hansen and Sargent (2001) introduce two robust decision rules to model the situation where a decision maker facing uncertainty has an approximating probabilistic model and is also concerned about prior misestimation. The two rules are the constraint criterion and the multiplier criterion. Before giving their formulas, we first introduce the definition of relative entropy.

Given $p, q \in \Delta$, $p \ll q$ denotes that p is absolutely continuous with respect to q.

Definition 6. *Given $p, q \in \Delta$, the relative entropy $R(p\|q)$ of p with respect to q is defined by*

$$R(p\|q) = \begin{cases} \sum_S p_i \log \dfrac{p_i}{q_i} & \text{if } p \ll q \\ \infty & \text{otherwise.} \end{cases}$$

Relative entropy is a measure of the "difference" or "distance" between two probabilities. Note that $R(\cdot\|q) : \Delta \to [0, \infty]$ is a lower semicontinuous convex function satisfying that $R(p\|q) = 0$ if and only if $p = q$.

Definition 7. *A constraint representation of a preference* $\succsim$ *is a triple* $\langle u, q, \eta \rangle$ *such that*
(1) $u : X \to \mathbb{R}$ *is a non-constant affine utility function,* $q \in \Delta$ *is an approximating prior, and* $\eta \in [0, \infty)$ *is a parameter;*
(2) for $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = \min_{p \in \{p \in \Delta | R(p\|q) \leq \eta\}} E_p u(f), \ \forall f \in \mathcal{F}, \tag{10}$$

V represents $\succsim$.

A preference $\succsim$ *admitting a constraint representation is called a constraint preference.*

Constraint representation differs from variant constraint representation in three respects. First, the decision maker has a single best-guess prior q instead of a set of them. Second, the "distance" between two priors is measured by relative entropy instead of the Euclidean metric. Third, the distance constraint η is a constant rather than a function of utility profiles. The parameter η measures the degree of his concern with prior misspecification. Larger values of η correspond to less trust in q.

Because of the properties of relative entropy, $\{p \in \Delta | R(p\|q) \leq \eta\}$ is a non-empty closed convex set. Thus, constraint preferences are maxmin preferences with the constraint set being specified in a concrete way.

Definition 8. *A multiplier representation of a preference* $\succsim$ *is a triple* $\langle u, q, \theta \rangle$ *such that*
(1) $u : X \to \mathbb{R}$ *is a non-constant affine utility function,* $q \in \Delta$ *is an approximating prior, and* $\theta \in [0, \infty)$ *is a parameter;*
(2) for $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = \min_{p \in \Delta} [E_p(u(f)) + \theta R(p\|q)], \ \forall f \in \mathcal{F}, \tag{11}$$

V represents $\succsim$.

A preference $\succsim$ *admitting a multiplier representation is called a multiplier preference.*

The multiplier representation is a special variational representation with the cost function being $\theta R(p\|q)$. A decision maker with a multiplier preference has an approximating prior q. He considers each prior p, but this incurs a cost which is proportional to the "distance" between p and q. The parameter θ measures the degree of his concern with prior misspecification. In contrast to η in (10), Larger values of θ correspond to more trust in q. If $\theta = \infty$, then with the convention that $0 \cdot \infty = 0$, the decision maker evaluates each act by the expected utility with respect to q.

Hansen and Sargent (2001) establish the connection between constraint and multiplier criteria. They show in a dynamic resource allocation problem that under some conditions, for each η in (10), there exists a θ in (11) such that (11) implies the same optimal solution as (10), and vise versa. However, as preference orderings, constraint criterion and multiplier criterion give totally different rankings of acts other than the optimal one. We obtain a constraint type of representation for multiplier preferences, which clearly shows the relationship between the two criteria.

Given $f, g \in \mathcal{F}$ and $A \subseteq S$, define $f_A g$ to be an act in $\mathcal{F}$ such that $f_A g(s) = f(s)$ for all $s \in A$, and $f_A g(s) = g(s)$ for all $s \in S \setminus A$. An event $A \subseteq S$ is *nonnull* if there exist $f, g, h \in \mathcal{F}$ such that $f_A h \succ g_A h$.

Proposition 4. *Suppose that S has at least three nonnull events. Then the following two statements are equivalent.*

(1) A preference $\succsim$ admits a multiplier representation $\langle u, q, \theta \rangle$.

(2) Define $V : \mathcal{F} \to \mathbb{R}$ by

$$V(f) = \min_{p \in \{p \in \Delta | R(p\|q) \leq \eta(u(f))\}} E_p u(f), \ \forall f \in \mathcal{F},$$

where $\eta : u(X)^S \to \mathbb{R}_+$ is given by $\eta(u(f)) = \min_{p \in \{p \in \Delta | E_p u(f) = -\theta \log E_q e^{-\frac{u(f)}{\theta}}\}} R(p\|q)$ for all $f \in \mathcal{F}$. The function V represents $\succsim$.

This result provides an equivalent representation for multiplier preferences which can be directly compared with constraint representation. It shows that a multiplier preference also admit a "constraint representation" except that the "distance" constraint is a specific function of utility profiles rather than a fixed number. Thus, the multiplier criterion induces the changing robustness concern for different acts, while constraint criterion assumes the constant robustness concern for all acts.

In particular, the function η has three properties. Suppose without loss of generality that $u(X) = \mathbb{R}$. First, $\eta(\phi + t\mathbb{1}) = \eta(\phi)$ for all $\phi \in \mathbb{R}^S$ and $t \in \mathbb{R}$, which shows that there is no wealth effect for multiplier preferences. Second, $\eta(k\phi) \geq \eta(\phi)$ if $k \geq 1$, which shows that multiplier preferences display increasing relative uncertainty aversion property. Lastly, $\lim_{k \searrow 0} \eta(k\phi) = 0$. This means that when the utility scale diminishes to 0, the decision maker tends to consider only his best-guess prior q. Moreover, the later two properties in turn imply that a multiplier preference and a constraint preference coincide if and only if both are represented by the subjective expected utility $V(f) = E_q u(f)$ for all $f \in \mathcal{F}$.

This equivalent representation of multiplier preferences closely resembles the variant constraint representation of DAUA variational preferences. The only difference lies in the measure of "distance" between priors. In fact, the Euclidean distance in the variant constraint representation can be replaced with a general measures of "distance". Let $l : \Delta \times \Delta \to \mathbb{R}_+$ be a lower semicontinuous function such that $l(\cdot, q) : \Delta \to \mathbb{R}_+$ is convex in the first term, and $l(p, q) = 0$ if and only if $p = q$. Define $L : \Delta \times \{B \subseteq \Delta | B \neq \emptyset\} \to \mathbb{R}_+$ by $L(p, B) = \inf_{q \in B} l(p, q)$ for any $p \in \Delta$ and non-empty set $B \subseteq \Delta$. Then all such functions L can be used as a "distance" measure in the variant constraint representation for DAUA variational preferences which include multiplier preferences. These measures are equivalent in the sense that with the corresponding "distance" constraint functions η, they all induce the same evaluation of acts.

Although one can use the general "metric" as above to obtain another variant constraint representation for multiplier preferences, relative entropy cannot be used in general to derive the variant constraint representation of DAUA variational preferences. One feature of relative entropy is that it assigns ∞ to all the priors that are not absolutely continuous with respect to the central prior q. This implies that the decision maker believes for sure that the states to which q assigns zero probability would never happen, and he disregards all the priors that assign positive probability to those states. This is not generally true for DAUA variational preferences. A decision maker with a DAUA variational preference typically considers all kinds of perturbation of the central prior q.

6 Conclusion

This paper axiomatizes a class of preferences which display decreasing absolute uncertainty aversion. We obtain three equivalent representations: variant constraint representation, weighted maxmin representation, and DAUA variational representation. This class of preferences includes variational preferences as a subclass. When restricted to this subclass, the first two representations are equivalent to the established variational representation. Moreover, a constraint type of representation is obtained for multiplier preferences. This representation directly shows the relationship between multiplier and constraint preferences.

In closing, we remark that three representations can be similarly obtained for the analogous class of preferences with *increasing* absolute uncertainty aversion. The only difference is that when the baseline utility of an act rises, the distance constraint η and the weight function λ weakly increase, while the cost function c weakly decreases in utilities.

7 Appendix: proofs

We denote by $\mathbb{Z}_+$ the set of positive integers, Δ° the interior of Δ and $\partial\Delta$ the boundary of Δ. Let $I : \mathbb{R}^S \to \mathbb{R}$ be given. We say that I is *normalized* if $I(t\mathbb{1}) = t$ for all $t \in \mathbb{R}$. We say that I is *constant superadditive* if $I(\varphi + t\mathbb{1}) \geq I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \geq 0$. Similarly, I is said to be *constant additive* if $I(\varphi + t\mathbb{1}) = I(\varphi) + t$ for all $\varphi \in \mathbb{R}^S$ and $t \in \mathbb{R}$. If $I(\varphi + \varphi') \geq I(\varphi) + I(\varphi')$ for all $\varphi, \varphi' \in \mathbb{R}^S$, then I is said to be *superadditive*.

Lemma 1. *A preference $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1 - A.6 if and only if there exists an affine onto function $u : X \to \mathbb{R}$ and a functional $I : \mathbb{R}^S \to \mathbb{R}$ such that*
(1) it is normalized, weakly increasing, quasi-concave, continuous and constant superadditive;
(2) $f \succsim g \Leftrightarrow I(u(f)) \geq I(u(g))$ for all $f, g \in \mathcal{F}$.

Moreover, u is unique up to a positive affine transformation, and given u, there is a unique normalized functional $I : \mathbb{R}^S \to \mathbb{R}$ such that the above condition (2) holds.

Proof. The necessity is easy. For the sufficed, the existence and uniqueness of the required u and I follow from Maccheroni, Marinacci and Rustichini (2006) (Lemma 28), Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) (Lemma 57), and Kopylov (2001), except that I is constant superadditive. We check now the constant superadditivity of I.

Let $\varphi \in \mathbb{R}^S$ and $t \geq 0$ be given. Suppose $x, x_0 \in X$ and $f \in \mathcal{F}$ such that $u(x) = 2t$, $u(x_0) = 0$ and $u(f) = 2\varphi$. Then $u(\frac{1}{2}f + \frac{1}{2}x) = \varphi + t\mathbb{1}$ and $u(\frac{1}{2}f + \frac{1}{2}x_0) = \varphi$. Since u is an affine onto function, then there exists $z \in X$ such that $\frac{1}{2}f + \frac{1}{2}x_0 \sim \frac{1}{2}z + \frac{1}{2}x_0$ for some $z \in X$. Since

$t \geq 0$, then $x \succsim x_0$. By Axiom A.2, we know that $\frac{1}{2}f + \frac{1}{2}x \succsim \frac{1}{2}z + \frac{1}{2}x$. Thus,

$$
\begin{aligned}
&I(\varphi + t\mathbb{1}) = I(u(\frac{1}{2}f + \frac{1}{2}x)) \geq I(u(\frac{1}{2}z + \frac{1}{2}x)) \\
&= \frac{1}{2}u(z) + \frac{1}{2}u(x) = \frac{1}{2}u(z) + \frac{1}{2}u(x_0) + \frac{1}{2}u(x) = u(\frac{1}{2}z + \frac{1}{2}x_0) + t \\
&= I(u(\frac{1}{2}f + \frac{1}{2}x_0)) + t = I(\varphi) + t.
\end{aligned}
$$

Checking the equalities and inequalities above mainly use the fact that I is normalized and satisfies condition (2), and that u is affine. □

Proof of Theorem 1. We check the sufficiency first. Let $u : X \to \mathbb{R}$ and $I : \mathbb{R}^S \to \mathbb{R}$ be given as in Lemma 1. We define $J : \mathbb{R}^S \to \mathbb{R}$ by

$$J(\varphi) = \lim_{k \searrow 0} \lim_{t \to \infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t] \tag{12}$$

for all $\varphi \in \mathbb{R}^S$.

To check that $J : \mathbb{R}^S \to \mathbb{R}$ is well defined, we show some stronger results.

First, for all $k > 0$ and $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]$ is bounded in $[\min_S \varphi(s), \max_S \varphi(s)]$. This is because I is weakly increasing.

Second, fix $k > 0$, $\frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]$ weakly increases in t. Indeed, if $t' \geq t$, then by the constant supadditivity of I, $I(k\varphi + t'\mathbb{1}) - t' = I(k\varphi + t\mathbb{1} + (t' - t)\mathbb{1}) - t' \geq I(k\varphi + t\mathbb{1}) + t' - t - t' = I(k\varphi + t\mathbb{1}) - t$.

Third, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]$ weakly decreases in k when $k > 0$. Suppose for the sake of a contradiction that $k' \geq k > 0$ and $\lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi + t\mathbb{1}) - t] > \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]$. Hence, there exists $\bar{t}$ such that for all $t, t' \geq \bar{t}$, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t] > \lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi + t\mathbb{1}) - t]$. That is,

$$I(k\varphi + t\mathbb{1}) < \frac{k}{k'}I(k'\varphi + t'\mathbb{1}) + t - \frac{k}{k'}t'. \tag{13}$$

Pick $t, t' \geq \bar{t}$ such that

$$\frac{k}{k'}t' + (1 - \frac{k}{k'})I(k'\varphi + t'\mathbb{1}) = t. \tag{14}$$

Thus, $k\varphi + t\mathbb{1} = \frac{k}{k'}(k'\varphi + t'\mathbb{1}) + (1 - \frac{k}{k'})I(k'\varphi + t'\mathbb{1})$. Since I is normalized and quasi-concave, then

$$I(k\varphi + t\mathbb{1}) \geq \frac{k}{k'}I(k'\varphi + t'\mathbb{1}) = \frac{k}{k'}I(k'\varphi + t'\mathbb{1}) + t - \frac{k}{k'}t',$$

where the equality follows from the choice of t and t'. This is a contradiction to (13) as desired.

The three results above guarantee that $J : \mathbb{R}^S \to \mathbb{R}$ is well defined. Note that $I(\varphi) \leq J(\varphi)$ for all $\varphi \in \mathbb{R}^S$.

We further show some properties of J.

It is easy to see that J is normalized and weakly increasing. It directly follows from the same properties of I.

Besides, J is constant additive. Let $\varphi \in \mathbb{R}^S$ and $r \in \mathbb{R}$ be given. Then

$$\begin{aligned} J(\varphi + r\mathbb{1}) &= \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k(\varphi + r\mathbb{1}) + t\mathbb{1}) - t] \\ &= \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + (kr+t)\mathbb{1}) - (kr+t) + kr] \\ &= \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + (kr+t)\mathbb{1}) - (kr+t)] + r \\ &= \lim_{k\searrow 0}\lim_{t'\to\infty} \frac{1}{k}[I(k\varphi + t'\mathbb{1}) - t'] + r = J(\varphi) + r, \end{aligned}$$

as desired.

Moreover, J is positive homogeneous of degree 1. Let $\varphi \in \mathbb{R}^S$ and $l > 0$ be given. Then

$$\begin{aligned} J(l\varphi) &= \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(kl\varphi + t\mathbb{1}) - t] \\ &= \lim_{k\searrow 0}\lim_{t\to\infty} \frac{l}{kl}[I(kl\varphi + t\mathbb{1}) - t] = l\lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{kl}[I(kl\varphi + t\mathbb{1}) - t] \\ &= l\lim_{k'\searrow 0}\lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi + t\mathbb{1}) - t] = lJ(\varphi), \end{aligned}$$

as desired.

Lastly, J is superadditive. Suppose for the sake of a contradiction that $J(\varphi+\varphi') < J(\varphi)+J(\varphi')$ for some $\varphi, \varphi' \in \mathbb{R}^S$. Since J is positive homogeneous of degree 1, then $J(\frac{1}{2}\varphi + \frac{1}{2}\varphi') < \frac{1}{2}J(\varphi) + \frac{1}{2}J(\varphi')$. Thus, there exsit $k > 0$ and $\bar{t} \geq 0$ such that for all $t, t' \geq \bar{t}$ and $t'' \in \mathbb{R}$,

$$\frac{1}{k}[I(k(\frac{1}{2}\varphi + \frac{1}{2}\varphi') + t''\mathbb{1}) - t''] < \frac{1}{2k}[I(k\varphi + t\mathbb{1}) - t] + \frac{1}{2k}[I(k\varphi' + t'\mathbb{1}) - t']. \tag{15}$$

By rearranging the terms, we get

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' + t''\mathbb{1}) < \frac{1}{2}I(k\varphi + t\mathbb{1}) + \frac{1}{2}I(k\varphi' + t'\mathbb{1}) + t'' - \frac{t+t'}{2}. \tag{16}$$

Pick $t \geq \bar{t}$ such that $k\min_S \varphi(s) + t \geq k\max_S \varphi'(s) + \bar{t}$. Thus $I(k\varphi + t\mathbb{1}) \geq k\min_S \varphi(s) + t \geq k\max_S \varphi'(s) + \bar{t}$. Pick $t' \in \mathbb{R}$ such that $I(k\varphi + t\mathbb{1}) = I(k\varphi' + t'\mathbb{1})$. Thus $t' \geq \bar{t}$. Let $t'' = \frac{t+t'}{2}$ so that $\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' + t''\mathbb{1} = \frac{1}{2}(k\varphi + t\mathbb{1}) + \frac{1}{2}(k\varphi' + t'\mathbb{1})$. Since I is superadditive, then

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' + t''\mathbb{1}) \geq \frac{1}{2}I(k\varphi + t\mathbb{1}) + \frac{1}{2}I(k\varphi' + t'\mathbb{1})$$

which is a contradiction to (16), as desired.

Since the functional $J : \mathbb{R}^S \to \mathbb{R}$ satisfies the properties above, then by Gilboa and Schmeidler (1989)'s Lemma 3.5, there exists a non-empty closed convex set $B \subseteq \Delta$ such that for all $\varphi \in \mathbb{R}^S$, $J(\varphi) = \min_B E_p\varphi$.

Define a set $D(\varphi) \subseteq \{p \in \Delta | I(\varphi) = E_p\varphi\}$ for each $\varphi \in \mathbb{R}^S$. Note that $D(\varphi)$ is a non-empty compact set for all $\varphi \in \mathbb{R}^S$. Define $\eta : \mathbb{R}^S \to \mathbb{R}_+$ as $\eta(\varphi) = d(D(\varphi), B)$. We want to check that $I(\varphi) = \min_{\{p\in\Delta|d(p,B)\leq\eta(\varphi)\}} E_p\varphi$ for all $\varphi \in \mathbb{R}^S$.

Fix $\varphi \in \mathbb{R}^S$. Since $D(\varphi)$ is non-empty and compact, then by the definition of $\eta(\varphi)$, there exists $p_* \in D(\varphi)$ such that $d(p, B) = \eta(\varphi)$. Hence, $I(\varphi) \leq \min_{\{p\in\Delta|d(p,B)\leq\eta(\varphi)\}} E_p\varphi$. Now we check that for all $p \in \Delta$ with $d(p, B) \leq \eta(\varphi)$, $E_p\varphi \geq I(\varphi)$. Suppose for the sake of a contradiction that there exists $p' \in \Delta$ such that $d(p', B) \leq \eta(\varphi)$ and $E_{p'}\varphi < I(\varphi)$. Denote by q a prior in B such that $d(p', q) = d(p', B)$. Then $E_q\varphi \geq \min_B E_p\varphi \geq I(\varphi) > E_{p'}\varphi$. Thus there exists $\alpha \in [0, 1)$ such that $I(\varphi) = E_{\alpha p'+(1-\alpha)q}\varphi$, which means that $\alpha p' + (1 - \alpha)q \in D(\varphi)$. If $d(p', q) > 0$, then

$$\begin{aligned} d(D(\varphi), B) &\leq d(\alpha p' + (1 - \alpha)q, q) = \alpha d(p', q) \\ &< d(p', q) = d(p', B) \leq \eta(\varphi) \end{aligned}$$

which contradicts the definition of η. If $d(p', q) = 0$, then $p' = q$, and thus

$$E_q\varphi = E_{p'}\varphi < I(\varphi) \leq \min_B E_p\varphi$$

which is again a contradiction since $q \in B$.

Moreover, for all $p \in \Delta$ with $d(p, B) < \eta(\varphi)$, $E_p\varphi > I(\varphi)$. Indeed, if $d(p', B) < \eta(\varphi)$ and $E_{p'}\varphi = I(\varphi)$ for some $\varphi \in \mathbb{R}^S$ and $p' \in \Delta$, then $p' \in \mathcal{D}(\varphi)$. Thus, $\eta(\varphi) = \min_{D(\varphi)} d(p, B) \leq d(p', B) < \eta(\varphi)$, which is a contradiction.

Now we check the properties of $\eta : \mathbb{R}^S \to \mathbb{R}_+$.

First, η is continuous on $\mathbb{R}^S \setminus \mathbb{R}\mathbb{1}$. Since $\eta(\varphi) = \min_{D(\varphi)} d(p, B)$ for all $\varphi \in \mathbb{R}^S$, by the maximum theorem, it suffices to check that $D : \mathbb{R}^S \rightrightarrows \Delta$ is continuous as a correspondence. Fix an arbitrary $\varphi \in \mathbb{R}^S$, and we check that D is upper hemicontinuous at φ. Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. Let $\{p_n\}_{n=1}^\infty$ be a sequence of elements in Δ such that $\lim_{n\to\infty} p_n = p$ for some $p \in \Delta$. Then $E_p\varphi = \lim_{n\to\infty} E_{p_n}\varphi_n = \lim_{n\to\infty} I(\varphi_n) = I(\lim_{n\to\infty} \varphi_n) = I(\varphi)$. Thus $p \in \mathcal{D}(\varphi)$.

Fix an arbitrary $\varphi \in \mathbb{R}^S \setminus \mathbb{R}\mathbb{1}$, and D is lower hemicontinuous at φ. To see it, let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. Fix $p \in D(\varphi)$. Suppose that $p \in$

Δ°. Let $\epsilon > 0$ be arbitrarily given. Define $A(\epsilon) = \{q \in \Delta | d(q,p) \le \epsilon\}$. Thus $E_p\varphi \in (\min_{A(\epsilon)} E_q\varphi, \max_{A(\epsilon)} E_q\varphi)$. There exists $N \in \mathbb{Z}_+$ such that $I(\varphi_n) \in (\min_{A(\epsilon)} E_q\varphi_n, \max_{A(\epsilon)} E_q\varphi_n)$ whenever $n \ge N$. Therefore, there exists $p_n \in A(\epsilon)$ for each $n \ge N$ such that $I(\varphi_n) = E_{p_n}\varphi_n$. Now for each $j \in \mathbb{Z}_+$, pick $p_{n_j} \in A(\frac{1}{j})$ such that $p_{n_j} \in D(\varphi_{n_j})$ and $n_{j'} > n_j$ for all $j' > j$ in $\mathbb{Z}_+$. Thus we get a sequence $\{p_{n_j}\}_{j=1}^\infty$ such that $p_{n_j} \in D(n_j)$ for all $j \in \mathbb{Z}_+$ and $\lim_{j\to\infty} p_{n_j} = p$. Now if $p \in \partial\Delta$, one can find such a sequence via a sequence $\{p_m\}_{m=1}^\infty$ of elements in Δ° that converges to p.

Second, $\eta(\varphi + t\mathbb{1})$ weakly decreases in $t \in \mathbb{R}$ for all $\varphi \in \mathbb{R}^S$. Let $\varphi \in \mathbb{R}^S$ and $t \le t'$ in $\mathbb{R}$ be given. For all $p \in D(\varphi + t\mathbb{1})$, $E_p[\varphi + t'\mathbb{1}] = I(\varphi + t\mathbb{1}) + t' - t \le I(\varphi + t'\mathbb{1})$. Suppose that $\eta(\varphi + t\mathbb{1}) = d(p', B)$ for some $p' \in \mathcal{D}(\varphi + t\mathbb{1})$. If $E_{p'}[\varphi + t'\mathbb{1}] < I(\varphi + t'\mathbb{1})$, then from the analysis above we know that $d(p', B) > \eta(\varphi + t'\mathbb{1})$, and thus $\eta(\varphi + t\mathbb{1}) > \eta(\varphi + t'\mathbb{1})$. If $E_{p'}[\varphi + t'\mathbb{1}] = I(\varphi + t'\mathbb{1})$, then $p' \in \mathcal{D}(\varphi + t'\mathbb{1})$, and thus $\eta(\varphi + t\mathbb{1}) = d(p', B) \ge \min_{D(\varphi + t'\mathbb{1})} d(p, B) = \eta(\varphi + t'\mathbb{1})$.

Third, for any $\varphi \in \mathbb{R}^S$, $\lim_{t\to\infty} \eta(k\varphi + t\mathbb{1})$ weakly increases in k when $k > 0$. We first show that it is equivalent to check that $\min_{\{q\in\Delta | \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]\}} d(q, B)$ weakly increased in k when $k > 0$. Let $\varphi \in \mathbb{R}^S$ and $k > 0$ be given. Define $T : [0, \infty] \rightrightarrows \Delta$ as a correspondence by $T(t) = \{p \in \Delta | E_p\varphi = \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]\}$ for each $t \in \mathbb{R}$, and $T(\infty) = \{p \in \Delta | E_p\varphi = \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]\}$. Note that for all $t \in [0, \infty]$, $T(t)$ is non-empty and compact. We show first that $\min_{T(\infty)} d(p, B) = \lim_{t\to\infty} \theta(k\varphi + t\mathbb{1})$. To do it, we check that T is continuous at ∞. For the upper hemicontinuity, let $\{t_n\}_{n=1}^\infty$ be a sequence of elements in $[0, \infty]$ such that $\lim_{n\to\infty} t_n = \infty$, and let $\{p_n\}_{n=1}^\infty$ be a sequence of elements in Δ such that $p_n \in T(t_n)$ for all $n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} p_n = p$ for some $p \in \Delta$. Then $E_p\varphi = \lim_{n\to\infty} E_{p_n}\varphi = \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]$. Hence, $p \in T(\infty)$. For the lower hemicontinuous, let $\{t_n\}_{n=1}^\infty$ be a sequence of elements in $[0, \infty]$ such that $\lim_{n\to\infty} t_n = \infty$. Fix $p \in T(\infty)$. Suppose that $q \in \Delta$ and $E_q\varphi = \min_S \varphi(s)$. If $E_p\varphi = E_q\varphi$, then $p \in T(t_n)$ for all $n \in \mathbb{Z}_+$ and thus we are done. Suppose that $E_p\varphi > E_q\varphi$. Then for each $n \in \mathbb{Z}_+$, there exists a unique $\alpha_n \in [0, 1]$ such that $\alpha_n p + (1 - \alpha_n)q \in T(t_n)$. Therefore, $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{1}{E_p\varphi - E_q\varphi}[\frac{1}{k}I((k\varphi + t_n\mathbb{1}) - t_n) - E_q\varphi] = \frac{1}{E_p\varphi - E_q\varphi}(E_p\varphi - E_q\varphi) = 1$. Thus, $\lim_{n\to\infty}[\lambda_n p + (1 - \lambda_n)q] = p$. Hence, by the maximum theorem, $\min_{T(infty)} d(q, B) = \lim_{t\to\infty} \min_{T(t)} d(q, B) = \lim_{t\to\infty} \min_{E_q(k\varphi + t\mathbb{1}) = I(k\varphi + t\mathbb{1})} d(q, B) = \lim_{t\to\infty} \eta(k\varphi + t\mathbb{1})$.

Keep $\varphi \in \mathbb{R}^S$ being fixed. Let $k' \ge k > 0$ be given. Let T be defined as above for k, and T' be analogously defined for k'. Suppose that $p \in T(\infty)$ and $d(p, B) = \min_{T(\infty)} d(q, B)$. Similarly, suppose that $p' \in T'(\infty)$ and $d(p, B) = \min_{T'(\infty)} d(q, B)$. We would like to check that $d(p, B) \le d(p', B)$. Note that $E_p\varphi \ge E_{p'}\varphi$. If $E_p\varphi = E_{p'}\varphi$, then $p' \in T(\infty)$, and thus $\min_{T(\infty)} d(q,B) \leq d(p',B) = \min_{T'(\infty)} d(q,B)$. If $E_p\varphi > E_{p'}\varphi$, then pick $q' \in B$ such that $d(p',B) = d(p',q')$. Since $E_{q'}\varphi \geq J(\varphi) \geq E_p\varphi$, then there uniquely exists $\alpha \in [0,1)$ such that $E_{\alpha p'+(1-\alpha)q'}\varphi = E_p\varphi$, i.e., $\alpha p' + (1-\alpha)q' \in T(\infty)$. Thus $d(p,B) \leq d(\alpha p' + (1-\alpha)q', B) \leq d(\alpha p' + (1-\alpha)q', q') = \alpha d(p',q') < d(p',q') = d(p',B)$, as desired.

Fourth, $\lim_{k\searrow 0}\lim_{t\to\infty} \eta(k\varphi + t\mathbb{1}) = 0$. Fix $\varphi \in \mathbb{R}^S$. Let $p \in B$ and $p' \in \Delta$ be given such that $E_p\varphi = J(\varphi)$ and $E_{p'}\varphi = \min_S \varphi(s)$. If $E_p\varphi = E_{p'}\varphi$, then for all $k > 0$ and $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi + t\mathbb{1}) - t] = E_p\varphi$. That is, for all $k > 0$ and $t \in \mathbb{R}$, $p \in D(k\varphi + t\mathbb{1})$, and thus $\min_{D(k\varphi+t\mathbb{1})} d(q,B) = 0$ since $p \in B$ as well. Hence, $\lim_{k\searrow 0}\lim_{t\to\infty} \eta(k\varphi + t\mathbb{1}) = 0$. If $E_p\varphi > E_{p'}\varphi$, then for each $k > 0$ there exists a unique $\alpha_k \in [0,1]$ such that $E_{\alpha_k p' + (1-\alpha_k)p}\varphi = \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t]$. Thus $\lim_{k\searrow 0} \alpha_k = \lim_{k\searrow 0} \frac{1}{E_{p'}\varphi - E'_p\varphi}[\lim_{t\to\infty} \frac{1}{k}(I(k\varphi + t\mathbb{1}) - t) - E_p\varphi] = \frac{E_p\varphi - E_p\varphi}{E_{p'}\varphi - E_p\varphi} = 0$. Combining the results above, we have that for each $k > 0$,

$$
\begin{aligned}
0 &\leq \lim_{k\searrow 0}\lim_{t\to\infty} \eta(k\varphi + t\mathbb{1}) \leq \lim_{t\to\infty} \eta(k\varphi + t\mathbb{1}) \\
&= \min_{\{q\in\Delta | \lim_{t\to\infty} \frac{1}{k}[I(k\varphi+t\mathbb{1})-t]\}} d(q,B) \leq d(\alpha_k p' + (1-\alpha_k p), p) = \alpha_k d(p',p).
\end{aligned}
$$

By taking the limit, we the desired result.

For necessity, we only check A.3, others are easy. Suppose that $\langle u, B, \eta \rangle$ is a variant constraint representation of $\succsim$ satisfying the conditions in Theorem 1(2). Let $I : \mathbb{R}^S \to \mathbb{R}$ be defined as in (6). It suffices to show that I is continuous. Let $\varphi \in \mathbb{R}^S$ be given. Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. If φ is constant, then

$$
\begin{aligned}
&\lim_{n\to\infty} |I(\varphi_n) - I(\varphi)| \leq \lim_{n\to\infty} \sup_S |\varphi_n(s) - I(\varphi)| \\
&= \lim_{n\to\infty} \sup_S |\varphi_n(s) - \varphi(s)| = 0,
\end{aligned}
$$

as desired. Suppose that φ is not constant. Define $W : R^S \rightrightarrows \Delta$ as a correspondence by $W(\phi) = \{q \in \Delta | d(q,B) \leq \eta(\phi)\}$ for all $\phi \in \mathbb{R}^S$. It suffices to show that W is continuous at φ. For upper hemicontinuity, let $\{p_n\}_{n=1}^\infty$ be a sequence of elements in Δ such that for each $n \in \mathbb{Z}_+$, $p_n \in W(\varphi_n)$ and $\lim_{n\to\infty} p_n = p$ for some $p \in \Delta$. Since η is continuous at φ, then $\lim_{n\to\infty} \eta(\varphi_n) = \eta(\varphi)$. Thus $d(p,B) = \lim_{n\to\infty} d(p_n,B) \leq \lim_{n\to\infty} \eta(\varphi_n) = \eta(\varphi)$. Hence, $p \in W(\varphi)$. For lower hemicontinuity, let $p \in W(\varphi)$ be given. If $d(p,B) < \eta(\varphi)$, then $p \in W(\varphi_n)$ for all sufficiently large n, and we are done. Suppose that $d(p,B) = \eta(\varphi)$. If $p \in B$, then $p \in W(\varphi_n)$ for all n, and we are done. If $p \notin B$, then $\eta(\varphi) > 0$. Suppose that $p' \in B$ and $d(p,p') = \eta(\varphi)$. Let $\{\epsilon_j\}_{j=1}^\infty$ be a sequence of real numbers such that for all $j \in \mathbb{Z}_+$, $\epsilon_j \in (0, \eta(\varphi))$ and $\lim_{j\to\infty} \epsilon_j = 0$. For each $j \in \mathbb{Z}_+$, pick $n_j \in \mathbb{Z}_+$ such that $|\eta(\varphi_{n_j}) - \eta(\varphi)| < \epsilon_j$ and $n_{j'} > n_j$ for all $j' > j$ in $\mathbb{Z}_+$. Define $\alpha_j = 1 - \frac{\epsilon_j}{\eta(\varphi)}$ for all $j \in \mathbb{Z}_+$. Then $\alpha_j \in (0,1)$ for all $j \in \mathbb{Z}_+$ and $\lim_{j\to\infty} \alpha_j p + (1-\alpha_j)p' = p$. For all $j \in \mathbb{Z}_+$, $d(\alpha_j p + (1-\alpha_j)p', B) \leq \alpha_j d(p, p') = \eta(\varphi) - \epsilon_j < \eta(\varphi_{n_j})$, and thus $\alpha_j p + (1-\alpha_j)p' \in W(\varphi_{n_j})$.

Lastly, for the uniqueness of the representation, let $< u', B', \eta' >$ be another variant constraint representation of $\succsim$ satisfying Theorem 1(2). Let $I' : \mathbb{R}^S$ be defined as in (6). Since both u and u' are affine functions representing preferences on constant acts. Then there exist $a > 0$ and $b \in \mathbb{R}$ such that $u'(x) = au(x) + b$ and $I'(a\varphi + b\mathbb{1}) = aI(\varphi) + b$ for all $x \in X$ and $\varphi \in \mathbb{R}^S$. Suppose for the sake of a contradiction that $B \neq B'$. Suppose further without loss of generality that $p \in B \setminus B'$. Then by a standard separation theorem, there exists $\varphi \in \mathbb{R}^S \setminus \mathbb{R}\mathbb{1}$ such that $E_p\varphi < \min_{B'} E_q\varphi$. Because of the properties of η, we have

$$\begin{aligned}
&\lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t] = \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[\min_{\{q\in\Delta | d(q,B)\leq \eta(k\varphi + t\mathbb{1})\}} E_p(k\varphi + t\mathbb{1}) - t] \\
&= \lim_{k\searrow 0}\lim_{t\to\infty} \min_{\{q\in\Delta | d(q,B)\leq \eta(k\varphi + t\mathbb{1})\}} = \min_{B} E_q\varphi < \min_{B'} E_q\varphi.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbb{1}) - t] = \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[\frac{I'[ak\varphi + (at+b)\mathbb{1}]}{a} - t] \\
&= \lim_{k\searrow 0}\lim_{t\to\infty} \min_{\{q\in\Delta | d(q,B')\leq \eta'(ak\varphi + (at+b)\mathbb{1})\}} = \min_{B'} E_q\varphi
\end{aligned}$$

which is a contradiction. Hence, $B = B'$.

For the uniqueness of distance bound, let $\varphi \in \mathbb{R}^S$ and $p, p' \in \Delta$ be given such that $I(\varphi) = E_p\varphi > \min_S \varphi(s) = E_{p'}\varphi$, and $d(p, B) \leq \eta(\varphi)$. Suppose without loss of generality that $\eta(\varphi) < \eta'(a\varphi + b)$. Then there exists $\epsilon \in (0,1)$ such that $d(\epsilon p + (1-\epsilon)p', B) \leq \eta'(a\varphi + b)$. Thus $I'(a\varphi + b) \leq E_{\epsilon p + (1-\epsilon)p'}[a\varphi + b] < aI(\varphi) + b$, which is a contradiction as desired.

Proof of Theorem 2. The idea of the proof is similar to that of Theorem 1. We only check the sufficiency, the necessity and uniqueness are easy. Let $u : X \to \mathbb{R}$ and $I : \mathbb{R}^S \to \mathbb{R}$ be given as in Lemma 1. We define $J : \mathbb{R}^S \to \mathbb{R}$ by

$$J(\varphi) = \lim_{k\to\infty}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbb{1}) + t]$$

for all $\varphi \in \mathbb{R}^S$.

Note that for all $\varphi \in \mathbb{R}^S$, $k > 0$ and $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi - t\mathbb{1}) + t] \in [\min_S \varphi(s), \max_S \varphi(s)]$. For all $\varphi \in \mathbb{R}^S$ and $k > 0$, $\frac{1}{k}[I(k\varphi - t\mathbb{1}) + t]$ weakly decreases in t. Moreover, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbb{1}) + t]$ weakly decreases in k. To see that, let $k' \geq k > 0$ be given. Suppose for the sake of a contradiction that $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbb{1}) + t] < \lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi - t\mathbb{1}) + t]$. Thus there exists $\bar{t}$ such that for all $t, t' \geq \bar{t}$, $\frac{1}{k}[I(k\varphi - t\mathbb{1}) + t] < \frac{1}{k'}[I(k'\varphi - t'\mathbb{1}) + t']$, that is,

$$I(k\varphi - t\mathbb{1}) < \frac{k}{k'}I(k'\varphi - t'\mathbb{1}) + \frac{k}{k'}t' - t. \tag{17}$$

Pick $t, t' \geq \bar{t}$ such that $\frac{k}{k'}t' + (\frac{k}{k'} - 1)I(k'\varphi - t'\mathbb{1}) = t$. Thus $k\varphi - t\mathbb{1} = \frac{k}{k'}I(k'\varphi - t'\mathbb{1}) + (1 - \frac{k}{k'})I(k'\varphi - t'\mathbb{1})\mathbb{1}$. Since I is quasi-concave and normalized, then $I(k\varphi - t\mathbb{1}) \geq \frac{k}{k'}I(k'\varphi - t'\mathbb{1}) + (1 - \frac{k}{k'})I(k'\varphi - t'\mathbb{1}) = \frac{k}{k'}I(k'\varphi - t'\mathbb{1})$, which contradicts (17). The above properties imply that J is well defined.

Similar to the proof of Theorem 1, J is normalized, weakly increasing, constant additive, positive homogeneous of degree 1 and superadditive. We only check superadditivity. Suppose the contrary that there exist $\varphi, \varphi' \in \mathbb{R}^S$, $J(\varphi + \varphi') < J(\varphi) + J(\varphi')$. Thus $J(\frac{1}{2}\varphi + \frac{1}{2}\varphi') < \frac{1}{2}J(\varphi) + \frac{1}{2}J(\varphi')$. Hence there exists $k > 0$ and $\bar{t} \in \mathbb{R}$ such that for all $t, t' \in \mathbb{R}$ and $t'' \geq \bar{t}$, $\frac{1}{k}[I(k(\frac{1}{2}\varphi + \frac{1}{2}\varphi') - t''\mathbb{1}) + t''] < \frac{1}{2k}[I(k\varphi - t\mathbb{1}) + t] + \frac{1}{2k}[I(k\varphi - t'\mathbb{1}) + t']$. Rearranging the terms, we get that

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbb{1}) < \frac{1}{2}I(k\varphi - t\mathbb{1}) + \frac{1}{2}I(k\varphi' - t'\mathbb{1}) + \frac{t + t'}{2} - t''. \tag{18}$$

Pick $t, t' \geq \bar{t}$ such that $I(k\varphi - t\mathbb{1}) = I(k\varphi' - t'\mathbb{1})$. Define $t'' = \frac{t+t'}{2}$. Note that $t'' \geq \bar{t}$ and $\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbb{1} = \frac{1}{2}(k\varphi - t\mathbb{1}) + \frac{1}{2}(k\varphi' - t'\mathbb{1})$. Since I is superadditive, then

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbb{1}) \geq \frac{1}{2}I(k\varphi - t\mathbb{1}) + \frac{1}{2}I(k\varphi' - t'\mathbb{1}),$$

which contradicts (18).

By Gilboa and Schmeidler (1989), there exists a unique non-empty closed convex set $C \subseteq \Delta$ such that $J(\varphi) = \min_C E_p\varphi$ for all $\varphi \in \mathbb{R}^S$. Fix $\varphi \in \mathbb{R}^S$. Then $I(\varphi) \in [\min_C E_p\varphi, \max_C E_p\varphi]$. We only check the upper bound. Let $t \in \mathbb{R}$ be given such that $I(\varphi) = I(-\varphi + t\mathbb{1})$. Since I is quasi-concave, then $I(\frac{\varphi}{2} + \frac{-\varphi+t\mathbb{1}}{2}) \geq \frac{1}{2}I(\varphi) + \frac{1}{2}I(-\varphi + t\mathbb{1}) \geq \frac{1}{2}I(\varphi) + \frac{1}{2}\min_C E_p(-\varphi) + \frac{t}{2}$. Since I is normalized, then $\frac{t}{2} \geq \frac{1}{2}I(\varphi) + \frac{1}{2}\min_C E_p(-\varphi) + \frac{t}{2}$. Thus, $I(\varphi) \leq -\min_C E_p(-\varphi) = \max_C E_p\varphi$.

Define $\lambda : \mathbb{R}^S \to [0, 1]$ by $\lambda(\varphi) = 1$ if $\min_C E_p\varphi = \max_C E_p\varphi$, and $\lambda(\varphi) = \frac{\max_C E_p\varphi - I(\varphi)}{\max_C E_p\varphi - \min_C E_p\varphi}$ otherwise. Note that $I(\varphi) = \lambda(\varphi)\min_C E_p\varphi + (1 - \lambda(\varphi))\max_C E_p\varphi$ for all $\varphi \in \mathbb{R}^S$.

Using the properties of I, it is easy to verify that $\lambda(\varphi + t\mathbb{1})$ weakly decreasing in t for all $\varphi \in \mathbb{R}^S$, $\lim_{t\to\infty} \lambda(k\varphi - t\mathbb{1})$ weakly increases in k when $k > 0$ for all $\varphi \in \mathbb{R}^S$, $\lim_{k\to\infty}\lim_{t\to\infty} \lambda(k\varphi - t\mathbb{1}) = 1$, and λ is continuous on $\{\varphi \in \mathbb{R}^S | \min_C E_p\varphi < \max_C E_p\varphi\}$.

Proof of Proposition 1. The proof mainly makes use of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011)'s representation theorems for uncertainty averse preferences. We first quote their result.

A.2′. Risk Independence. For all $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$x \sim y \Rightarrow \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

Theorem 3. [8] *The following two statements are equivalent.*

(1) A preference $\succsim$ *on* $\mathcal{F}$ *satisfies A.1, A.2′, A.3 - A.6.*

(2) There exists an affine onto function $u : X \to \mathbb{R}$*, and a lower semicontinuous function* $G : \mathbb{R} \times \Delta \to (-\infty, infty]$ *such that*

(i) G *is quasi-convex on* $\mathbb{R} \times \Delta$*;*

(ii) $G(\cdot, p)$ *is increasing for all* $p \in \Delta$*;*

(iii) $\inf_{\Delta} G(t, p) = t$ *for all* $t \in \mathbb{R}$*;*

(iv) for $I : \mathbb{R}^S \to \mathbb{R}$ *defined as*

$$I(\varphi) = \min_{\Delta} G(E_p\varphi, p) \; for \; all \; \varphi \in \mathbb{R}^S,$$

I *is continuous on* $\mathbb{R}^S$*; (v) for* $V : \mathcal{F} \to \mathbb{R}$ *defined as*

$$V(f) = I(u(f)) \; for \; all \; u \in \mathcal{F},$$

V *represents* $\succsim$.

Moreover, u *is unique up to a positive affine transformation. For each* u*,* G *is uniquely given by*

$$G(t, p) = \sup_{\mathcal{F}}\{u(x_f) : E_p u(f) \le t\} \; for \; all \; (t, p) \in \mathbb{R} \times \Delta. \tag{19}$$

For the sufficiency, suppose that $\succsim$ satisfies A.1 - A.6. Thus $\succsim$ satisfies A.2′ (see e.g. Lemma 28 of Maccheroni, Marinacci and Rustichini (2006)). Hence, there exists $u : X \to \mathbb{R}$ and $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ which satisfy the conditions in Theorem 3. Moreover, u is unique up to a positive affine transformation and G is uniquely given by (19).

Define $c : \mathbb{R} \times \Delta \to [0, \infty]$ by

$$c(t, p) = G(t, p) - t \; for \; all \; (t, p) \in \mathbb{R} \times \Delta.$$

[8] This theorem combines Theorem 3 and Theorem 5 in Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

We check that $< u, c >$ is a DAUA variational representation of $\succsim$. By definition and the corresponding properties of G, c is lower semicontinuous on $\mathbb{R}\times\Delta$, $c(t, p)+t$ is quasi-convex on $\mathbb{R}\times\Delta$, and $\inf_{\Delta} c(t, p) = 0$ for each $t \in \mathbb{R}$. Moreover, for I defined as in (9), I is continuous on $\mathbb{R}^S$, and $V : \mathcal{F} \to \mathbb{R}$ represents $\succsim$.

To see that $c(t, p)$ is weakly increasing in t, let $p \in \Delta$ and $t' \geq t$ in $\mathbb{R}$ be given. For any $f \in \mathcal{F}$ such that $E_p u(f) \leq t$, there exists $f' \in \mathcal{F}$ such that $u(f') = u(f) + (t' - t)\mathbb{1}$ and thus $E_p u(f') \leq t'$. Let I be given as in (9). Note that it is normalized and constant superadditive. Thus $u(x_{f'})-t' = I(u(f'))-t' = I(u(f)+t'-t)-t' \geq I(u(f))+t'-t-t' = I(u(f))-t = u(x_f)-t$. Since G is given by (19), then by definition $c(t', p) \geq c(t, p)$.

The necessity and uniqueness are routine. We only check the necessity for A.2. It suffices to show that I is constant superadditive. Fix $\varphi \in \mathbb{R}^S$, $p \in \Delta$ and $t > 0$. Then $I(\varphi + t\mathbb{1}) = \min_{\Delta}[E_p(\varphi+t\mathbb{1})+c(E_p(\varphi+t\mathbb{1}), p)] = \min_{\Delta}[E_p\varphi+c(E_p\varphi+t, p)]+t \geq \min_{\Delta}[E_p\varphi, c(E_p\varphi, p)]+t = I(\varphi) + t$, as desired.

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