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RESEARCH ARTICLE

Egalitarian division under Leontief Preferences

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Abstract We consider the problem of fairly dividing l divisible goods among n agents with the generalized Leontief preferences. We propose and characterize the class of generalized egalitarian rules which satisfy efficiency, group strategy-proofness, anonymity, resource monotonicity, population monotonicity, envy-freeness and consistency. On the Leontief domain, our rules generalize the egalitarian-equivalent rules with reference bundles. We also extend our rules to agent-specific and endowment-specific egalitarian rules. The former is a larger class of rules satisfying all the previous properties except anonymity and envy-freeness. The latter is a class of efficient, group strategy-proof, anonymous and individually rational rules when the resources are assumed to be privately owned.

Keywords Fair division · Egalitarian rules · Group strategy-proofness · Generalized Leontief preferences · Social choice · Exchange economies

JEL Classification D51 · D71

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1 Introduction

In the fair division literature, efficiency and strategy-proofness typically imply totally unfair outcomes. For example, Zhou (1991) shows that an efficient and strategy-proof allocation rule must be dictatorial already in a two-agent economy with continuous, strictly monotonic and strictly convex preferences. Such negative result has been extended to several more restricted domains by many researchers. Serizawa and Weymark (2003) further shows that in a many-agent many-good economy, no efficient and strategy-proof rule can guarantee every agent a consumption bundle bounded away from the origin. (Additional discussion of related literature is given at the end of this section.)

However, the picture changes a lot if we assume full complementarity among the goods and consider the domain of Leontief preferences. On the Leontief domain, for most efficient divisions of a given set of resources, some of the resources are redundant.[1] Thus, it makes sense to give the agents only the least amount of goods to achieve given welfare levels, while transferring the redundant resources to other potential users outside the rule.[2] We speak in this case of a *non-wasteful rule*. In addition to the normative concern of parsimony, the restriction to non-wasteful rules reduces the possibility of strategic manipulation. It turns out that then there exist rules satisfying efficiency, strategy-proofness and many fairness axioms.

The Leontief preferences and the corresponding non-wasteful rules are of natural practical interests, as shown in the computer science literature like Ghodsi et al. (2010), Hindman et al. (2011), Bodwin et al. (2011), Joe-Wong et al. (2011), Dolev et al. (2012), etc. For example, they consider multiple resource sharing problems in cloud computing systems. The users are allocated with computing resources like CPU, memory and I/O resources to do their different jobs with heterogeneous demands. In such circumstance, each user needs the resources in a customized proportion while redundant resources should not be allocated in order to avoid waste.

Two earlier papers inspire our work. Ghodsi et al. (2010) are the first to propose non-wasteful rules for the Leontief domain. They prove that in a many-agent many-good economy, the egalitarian-equivalent (EE) rule proposed by Pazner and Schmeidler (1978) (they call it the Dominant Resource Fairness mechanism) is efficient, strategy-proof and envy-free, and satisfies several other fairness axioms. Prior to them, Nicolò (2004) characterizes in a two-agent two-good economy with generalized Leontief preferences, a class of rules which are efficient, fully implementable in truthful strategies (a requirement stronger than strategy-proofness) and individually rational. However, Nicolò's rules are wasteful, and he finds it difficult to generalize his result to an economy with more agents and more goods.

[1] For example, in a two-agent two-good economy, both agents have the same preference represented by the utility function $u(x) = \min\{\frac{x_1}{2}, x_2\}$, and the endowment vector is (2, 2). Then, 1 unit of good 2 is redundant in any efficient allocation which divides up all the resources.

[2] Notice that here withholding the redundant resources does not affect efficiency since they are useless to the agents. It is different from the budget loss in VCG mechanisms which directly reduce the welfare of the agents.

Our contribution is to bring the existing results to a much more general level. Under Leontief preferences, we propose a class of non-wasteful rules which generalize the EE rules with reference bundles (see Sect. 3 for the relation of the EE rule and those with reference bundles). They satisfy efficiency, (group) strategy-proofness and almost all the fairness axioms in the literature (see below for further discussion). We also characterize our rules by these axioms. Moreover, the characterization works as well on a much larger preference domain—the generalized Leontief preference domain, which we shall discuss later. Lastly, we provide two natural extensions of our rules.

The rules we propose are called *generalized egalitarian rules* (defined in Sect. 3). A generalized egalitarian rule assumes that there is a continuous monotonic "benchmark preference" in the commodity space owned by the society. It looks for the non-wasteful efficient allocation where all the agents get the bundles among which the society is indifferent according to its benchmark preference. In another way, we can visualize that in the commodity space, the agents walk on their own "minimum-demand" paths associated with their Leontief preferences at some given speeds which guarantee that at any time they all simultaneously stand on the same indifference curve of the benchmark preference, and then our rule picks the end points where they reach the endowment feasibility constraints. Essentially, egalitarian rules set a standard for society to measure different ordinal preferences of the agents so that they are treated equally by this standard. While a classical EE rule makes the agents feel indifferent between their allocations and the same fraction of the social endowment, our rule gives the agents "equal" bundles according to a utility function of the society. It turns out that when the social endowment is fixed, a classic EE rule on the Leontief domain is one of our rules with a particular benchmark preference. We discuss about it in detail in Example 2 of Sect. 3.

There is another interpretation of generalized egalitarian rules. Thomson (1994) proposes a concept of equity to capture the notion of equal opportunities. Given a family $\mathcal{C}$ of choice sets, he defines an equal opportunity allocation relative to $\mathcal{C}$ as one giving every agent his optimal bundle from a common choice set in $\mathcal{C}$. Since such an allocation is obtained by having the agents choose in a common choice set, they can be viewed to get equal opportunities. It turns out that a general egalitarian rule always picks the Pareto-optimal equal opportunity allocation relative to a corresponding family of nested choice sets.

Our first main result (Theorem 1) shows that a generalized egalitarian rule satisfies efficiency, group strategy-proofness, anonymity, resource monotonicity, population monotonicity, envy-freeness and consistency; and conversely, given an efficient, resource monotonic and consistent rule, if it is either strategy-proof and anonymous, or envy-free, then it must be a generalized egalitarian rule.

All these axioms are very familiar in the fair division literature. Among the incentive compatibility axioms, group strategy-proofness is a very strong one. It allows no group of agents to misreport their preferences together and achieve Pareto improvement within the group (see Pattanaik 1978; Barberà 1979; Moulin and Shenker 2001; Serizawa 2006; Juarez 2008). For the fairness axioms, anonymity simply rules out the discrimination of the agents by their names; resource monotonicity guarantees that every agent benefits from the growth of the social endowment (see Roemer 1986a,b; Chun and Thomson 1988); population monotonicity ensures that no agent will get worse off when less agents join in the division (see Thomson 1983); and envy-freeness makes every agent weakly prefer his own allocation to anybody else's (see Foley 1967; Varian 1974, 1976). The consistency axiom has also played an important role in the fair division literature, in particular, the rationing (or bankruptcy) problems (see Aumann and Maschler 1985; Young 1987; Thomson 1988). It requires that when some agents leave first with their allocated bundles, if we apply the rule again to the reduced economy, the rest of the agents will still be allocated with the same bundles as in the original economy. For a survey of these and some other axioms in the fair division literature, see Thomson (2010).

Many of the axioms above are known to be very demanding and typically incompatible. For example, Moulin and Thomson (1988) show that any efficient and resource monotonic rule must generate envy in an economy with continuous, monotonic, convex and homothetic preferences. However, generalized egalitarian rules under Leontief preferences surprisingly satisfy them all.

Our rules and characterization apply for a much larger preference domain—the domain of generalized Leontief preferences (see Theorem 2). While for a standard Leontief preference, the set of minimum commodity bundles that achieve given utility levels, which we called the *critical set*, is a ray from the origin in the commodity space, the critical set of a generalized Leontief preference can be an arbitrary strictly increasing curve starting from the origin. In real life, generalized Leontief preferences are relevant when the agents are production units and the goods are inputs. For example, a group of people are dividing some cotton, silk and lace to make clothes. They would like to use these materials in different proportions according to their own tastes. Given the precise combination of the materials to make some pieces of clothes, more material of one kind is useless, which captures the essence of a Leontief preference. Moreover, when the amount of all materials increases, one might be able to make a dress instead of a shirt which requires different proportion of materials. There might also exist different types of returns to scale which alter the input proportion. Hence, one's critical set is an increasing curve, as exhibited in the generalized Leontief preferences.

Our results crucially depend on the restriction to non-wasteful rules. We give an example in Sect. 3 showing that our results do not hold without this restriction. Our characterization is tight with respect to all the axioms.

Our next two results (Theorems 3, 4) extend the generalized egalitarian rules in two directions. First, instead of using one single benchmark preference to measure all agents' utilities, a rule may assign to each agent a personal welfare index and equalize their utilities according to these agent-specific welfare indices. This family of rules is a much larger and non-anonymous class. Naturally, we do not expect envy-freeness in this case. However, all the other good properties are preserved.

The second extension is motivated when the resources are assumed to be privately, rather than commonly, owned by the agents. A compelling requirement here is the voluntary participation of the agents in the social reallocation. This is ensured by the individual rationality axiom, which requires the allocation to an agent to be no worse (for this agent) than his initial endowment. In this case, we can set the welfare indices such that it is always an "equal treatment" allocation to give every agent the minimum bundle that provides him the same welfare level as his private endowment. The welfare indices then depend on the endowment profile. By slightly modifying the argument in Moulin and Thomson (1988), one can check that efficiency, resource monotonicity and individual rationality are also incompatible in our context. We show that our endowment-specific egalitarian rules are efficient, group strategy-proof, anonymous, consistent and individually rational.

For both agent-specific and endowment-specific rules, our results are one-sided and we leave the characterizations as open questions.

After the literature review below, the paper is organized as follows. Section 2 presents the basic model and the axioms. Section 3 defines the generalized egalitarian rules under Leontief preferences and gives the characterization result. Section 4 introduces the generalized Leontief preference domain, on which the characterization still holds. Section 5 contains the main proofs. Section 6 checks the tightness of our characterization. Sections 7 and 8 provide two extensions of the generalized egalitarian rules: agent-specific and endowment-specific egalitarian rules. Section 9 provides concluding remarks. The "Appendix" contains some supporting proofs.

Related literature

For the incompatibility of efficiency and strategy-proofness with fairness properties in exchange economies, Hurwicz (1972) first proves that any efficient and individually rational rule is manipulable in two-agent two-good economies where both agents have continuous, strictly convex and strictly monotonic preferences. Dasgupta et al. (1979) replace individual rationality with non-dictatorship, while allowing discontinuous preferences. Zhou (1991) shows that in two-agent many-good exchange economies with the same preference domain as in Hurwicz (1972), a strategy-proof and efficient rule has to be inverse-dictatorial,[3] and hence dictatorial. From then on, many authors consider various restricted domains, either obtain similar impossibility results or compromise with weakened axioms, such as Schummer (1997, 2004), Ju (2003), Hashimoto (2008) and Momi (2011a) for two-agent cases, and Barberà and Jackson (1995), Kato and Ohseto (2002, 2004), Amorós (2002), Serizawa (2002), Serizawa and Weymark (2003), Ju (2004), Morimoto et al. (2012) and Momi (2011b) for many-agent cases. As we mentioned before, both Nicolò (2004) and Ghodsi et al. (2010) study the Leontief preference domain and achieve positive results. The main difference between their works is that Nicolò (2004) studies a two-agent two-good economy with generalized Leontief preferences and gives a characterization, while Ghodsi et al. (2010) study a many-agent many-good economy with standard Leontief preferences and give several one-sided results. In this paper, we consider generalized Leontief preferences and get very positive characterization results for many-agent many-good economy, without weakening any axioms. We would also like to mention that there is a large part of literature studying allocation rules for economies with public goods, such as Satterthwaite and Sonnenschein (1981); Hurwicz and Walker (1990); Schummer (1999); Serizawa (1999) and Moreno and Moscoso (2011).

[3] A rule is inverse-dictatorial if there exists some agent who always gets nothing. In a two-agent economy, it is equivalent to a dictatorial rule.

2 The model

Throughout this paper, for all $x, y \in \mathbb{R}^m$ where $m \in \mathbb{N}$, $x \geq y$ means that $x_k \geq y_k, \forall k = 1, \ldots, m$; $x > y$ means that $x_k > y_k, \forall k = 1, \ldots, m$. The latter will be the order that we refer to when we consider totally ordered sets in $\mathbb{R}^m$. Let $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x \geq 0\}$, $\mathring{\mathbb{R}}^m_+ = \{x \in \mathbb{R}^m | x > 0\}$, and $\partial\mathbb{R}^m_+ = \mathbb{R}^m_+ \setminus \mathring{\mathbb{R}}^m_+$. For any subsets S_1 and S_2 of $\mathbb{R}^m_+$, $S_1 + S_2 = \{s_1 + s_2 | s_1 \in S_1,\ s_2 \in S_2\}$, and similarly $S_1 - S_2 = \{s_1 - s_2 | s_1 \in S_1,\ s_2 \in S_2\}$.

Fix the set of perfectly divisible goods $L = \{1, \ldots, l\}$, $l \in \mathbb{N}$. Let $\mathbb{R}^l_+$ be the commodity space. Up to Sect. 3, every agent is assumed to have a standard Leontief preference on $\mathbb{R}^l_+$, which can be represented by a utility function $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$, $\forall x \in \mathbb{R}^l_+$, where x^k denotes the amount of the k-th good, $\lambda_k > 0$, $\forall k \in L$, and $\sum_{k \in L} \lambda_k = 1$ for normalization. Let $\mathcal{U}$ denote the set of all such utility functions.[4] We will generalize this preference domain in Sect. 4.

Definition 1 Let $u \in \mathcal{U}$ with $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$ be given. We call $\gamma = \{(\lambda_1 t, \ldots, \lambda_l t) \in \mathbb{R}^l_+ | t \in \mathbb{R}_+\}$ the critical set of the preference u.

A critical set of a preference $u \in \mathcal{U}$ consists of all the minimum commodity bundles required to achieve given utility levels. It is a ray starting from the origin and thus a connected, totally ordered and closed subset in $\mathbb{R}^l_+$. It is easy to see that γ is uniquely defined for each $u \in \mathcal{U}$. Hence, in the following, we will interchangeably use u and γ as needed.

An economy E is a triple (N, u_N, ω) where $N \subseteq \mathbb{N}$ is a nonempty finite set of agents, $u_N = (u_i)_{i \in N}$ with $u_i \in \mathcal{U}$, $\forall i \in N$, is a preference profile, and $\omega \in \mathbb{R}^l_+$ is the social endowment of the economy. Upto Sect. 7, the resources are assumed to be collectively owned. In Sect. 7, we consider the case where every agent has a private endowment and their endowments are put together to be divided. Let $\mathcal{E}$ denote the set of all economies.

Given (N, ω), the set of all feasible allocations is usually defined as $A(N, \omega) = \{x \in \mathbb{R}^{|N| \times l}_+ | \sum_{i \in N} x_i \leq \omega\}$, where x_i is the l dimensional bundle for agent i. We further require that the bundle of each agent is in his critical set. The reason is that the Leontief preferences are not strictly monotone, so society would like to keep the redundant goods in this economy for alternative use, in the spirit of non-wastefulness. Note that our main result does not hold when the allocations are allowed to be wasteful. A counter-example will be given at the end of Sect. 3.

Formally, for any economy $E = (N, u_N, \omega)$, we consider the restriction of $A(N, \omega)$ on the critical sets, $A^*(E) = A(N, \omega) \cap \prod_{i \in N} \gamma_i$ where γ_i is the critical set of u_i. Let $\mathcal{A}^* = \{A^*(E) | E \in \mathcal{E}\}$.

Definition 2 An allocation rule (or rule for simplicity) is a mapping $\mu : \mathcal{E} \rightarrow \mathcal{A}^*$ with $\mu(E) \in A^*(E)$, assigning to each economy a non-wasteful feasible allocation. For any $i \in N$, $\mu_i(E)$ denotes the bundle allocated to agent i.

[4] We normalize the utility functions so that our rules only care about the ordinal properties. However, it is not necessary for our result. It can be easily shown that any rule satisfying efficiency, strategy-proofness and consistency only takes into account the ordinal properties.

For notational simplicity, we write $\mu(u_N)$ (or $\mu(\omega)$) to denote $\mu(N, u_N, \omega)$, when (N, ω) (or (N, u_N)) is fixed.

Our normative requirements on rules are all very familiar in the literature (see the Introduction).

(I) Efficiency

Efficiency naturally requires that a rule always assigns Pareto-optimal allocations.

Given $E = (N, u_N, \omega)$, an allocation $x \in A(N, \omega)$ is *efficient* if there exists no $y \in A(N, \omega)$ such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$, and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. A rule μ is *efficient* (EFFN) if $\mu(E)$ is efficient for every $E \in \mathcal{E}$.

Lemma 1 *Given $E = (N, u_N, \omega)$, an allocation $x \in A^*(E)$ is efficient if and only if $\sum_{i \in N} x_i^k = \omega^k$ for some $k \in L$, where x_i^k denotes the amount of good k given to agent i.*

Proof For sufficiency, suppose the contrary that there exists $y \in A(N, \omega)$ such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$, and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. Then, $y_i \geq x_i$ for all $i \in N$ and $y_j > x_j$ for some $j \in N$, since $x_i \in \gamma_i$, $\forall i \in N$. Hence, $\sum_{i \in N} y_i > \sum_{i \in N} x_i$, and thus $\sum_{i \in N} y_i^k > \sum_{i \in N} x_i^k = \omega^k$, which contradicts feasibility. For necessity, suppose the contrary that $\sum_{i \in N} x_i < \omega$. Then, consider the allocation $y \in A(N, \omega)$ such that $y_i = x_i$, $\forall i \in N \setminus \{j\}$, and $y_j = x_j + \omega - \sum_{i \in N} x_i > x_j$. Clearly, it implies that x is not efficient, which is a contradiction. □

(II) Incentive compatibility

We require the familiar strategy-proofness and its strengthening as group strategy-proofness.

Let $\mathcal{U}_S = \mathcal{U}^{|S|}$, $\forall S \subseteq N$, and $\mathcal{U}_N$ is the set of all preference profiles. For any $S \subseteq N$, we denote by (u'_S, u_{-S}) the vector $u_N \in U_N$ with u_i replaced by u'_i, $\forall i \in S$. If $S = \{i\}$, we simply write (u'_i, u_{-i}).

A rule μ is *strategy-proof* (SP) if $\forall (N, u_N, \omega)$, $\forall i \in N$, $\forall u'_i \in \mathcal{U}$, $u_i(\mu_i(u_N)) \geq u_i(\mu_i(u'_i, u_{-i}))$.

A rule μ is *group strategy-proof* (GSP) if $\forall (N, u_N, \omega)$, there does not exist $S \subseteq N$ and $u'_S \in \mathcal{U}_S$ such that $u_i(\mu_i(u_N)) \leq u_i(\mu_i(u'_S, u_{-S}))$, $\forall i \in S$, and at least one inequality is strict.

(III) Fairness

There are four classic fairness axioms: anonymity, envy-freeness, resource monotonicity and population monotonicity. Envy-freeness and resource monotonicity are known to be very demanding and usually incompatible.

Let π be a bijection on $\mathbb{N}$. A rule μ is *anonymous* (ANON) if $\forall \pi$, $\forall (N, u_N, \omega)$, $\forall i \in N$, $\mu_i(N, u_N, \omega) = \mu_{\pi(i)}(\pi(N), (u_{\pi(j)})_{\pi(N)}, \omega)$ where $u_{\pi(j)} = u_j$, $\forall j \in N$.

Remark 1 If μ is ANON, then for any (N, u_N, ω) such that $u_i = u_j$, $i, j \in N$, $\mu_i(N, u_N, \omega) = \mu_j(N, u_N, \omega)$.

A rule μ is *envy-free* (EF) if $\forall (N, u_N, \omega)$, $\forall i, j \in N$, $u_i(\mu_i(N, u_N, \omega)) \geq u_i(\mu_j(N, u_N, \omega))$.

A rule μ is *resource monotonic* (RM) if $\forall (N, u_N)$, $\forall \omega, \omega' \in \mathbb{R}^l_+$, $\omega > \omega'$ implies that $u_i(\mu_i(\omega)) > u_i(\mu_i(\omega'))$, $\forall i \in N$.

There is another version of resource monotonicity. It states that $\forall (N, u_N)$, $\forall \omega$, $\omega' \in \mathbb{R}^l_+$, $\omega \geq \omega'$ implies that $u_i(\mu_i(\omega)) \geq u_i(\mu_i(\omega'))$, $\forall i \in N$. In general, these two versions do not imply each other. However, our rules below satisfy both of them, and the first one combined with the other axioms implies the second by our characterization result.

A rule μ is *population monotonic* (PM) if $\forall (N, u_N, \omega)$, $\forall N' \subseteq N$ and $N' \neq \emptyset$, $\forall i \in N'$, $u_i(\mu_i(N', u_{N'}, \omega)) \geq u_i(\mu_i(N, u_N, \omega))$.

(IV) Consistency

Consistency has played an important role in the rationing literature and also in the fair division problems of discrete goods.

A rule μ is *consistent* (CST) if $\forall (N, u_N, \omega)$, $\forall N' \subseteq N$ and $N' \neq \emptyset$, $\forall i \in N'$, $\mu_i(N, u_N, \omega) = \mu_i(N', u_{N'}, \omega - \sum_{j \in N \setminus N'} \mu_i(N, u_N, \omega))$.

Note that to check consistency, it is equivalent to check whether the corresponding condition holds when $|N'| = |N| - 1$.

Remark 2 It is easy to see that if a rule is consistent and resource monotonic (no matter which version of resource monotonicity is adopted), then it must be population monotonic. In the following, if a rule is CST and RM, we will just keep in mind that it is also PM without even mentioning in the theorems.

3 Generalized egalitarian rules

Let $f : \mathbb{D} \to \mathbb{R}^n$ where $\mathbb{D} \subseteq \mathbb{R}^m$ and $m, n \in \mathbb{N}$ be an arbitrary function. We say f is strictly increasing if $\forall x, y \in \mathbb{R}^m$, $x > y$ implies that $f(x) > f(y)$.

Suppose that $W : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ is a strictly increasing and continuous function. Given an economy $E = (N, u_N, \omega)$, let $A^W(E) = \{x \in A^*(E) | W(x_i) = W(x_j), \forall i, j \in N\}$.

Lemma 2 *$A^W(E)$ is a totally ordered and closed set in $\mathbb{R}^{|N| \times l}$. In particular,* $\max A^W(E)$ *exists.*

Proof To show that $A^W(E)$ is totally ordered, let $x, y \in A^W(E)$ such that $x \neq y$ be given. Suppose without loss of generality (WLOG) that $x_j < y_j$ for some $j \in N$. By the definition of $A^W(E)$ and the properties of W, we know that $\forall i \in N$, $x_i, y_i \in \gamma_i$, and $W(x_i) = W(x_j) < W(y_j) = W(y_i)$. Since the γ_i's are totally ordered sets and W is strictly increasing, then $x_i < y_i$, $\forall i \in N$, and thus, $x < y$.

To see that $\max A^W(E)$ exists, note that $A^*(E)$ is closed and W is continuous. Moreover, $A^W(E)$ is nonempty and bounded. Thus, $\max A^W(E)$ exists. □

Lemma 2 guarantees that the following rule is well defined.

Definition 3 A rule μ is called a generalized egalitarian rule, if there is a strictly increasing continuous function $W : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ such that for all $E \in \mathcal{E}$, $\mu(E) = \max A^W(E)$.

Let $\mathcal{M}$ denote the class of generalized egalitarian rules. We write μ^W when we want to indicate that μ is generated by W.

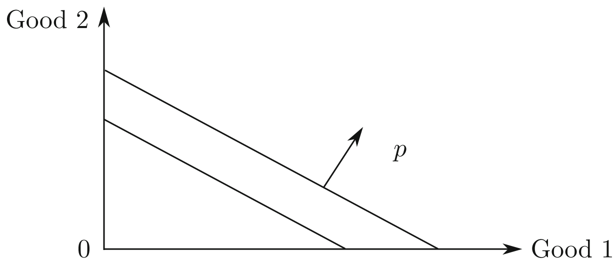

Fig. 1 Equalizing total wealth

We give two interpretations of our rules. One is in terms of a benchmark preference on the commodity space. The other is related to "equal opportunity allocations".

First, suppose that society has a benchmark preference over the commodity space which is represented by W.[5] Then, μ^W assigns to each agent the same welfare level according to this benchmark preference of society. We use two examples to explain.

Example 1 Equalizing total wealth.

Fix a price vector $p \in \mathring{\mathbb{R}}^l_+$. Let $W(x) = p \cdot x$, $\forall x \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$. In this case, society wants the agents to get the same total wealth. The indifference classes of the benchmark preference are just the budget lines. See Fig. 1 for an illustration in a two-good economy.

Example 2 Egalitarian-equivalent (EE) rules.

The spirit of the classic EE rule is that every agent should get "equal" share of the social endowment. The difficulty is to find a way of measuring these shares in a world of ordinal preferences (Moulin 1995). Pazner and Schmeidler (1978) were the first to propose a solution. It assigns an allocation at which the agents are indifferent between their bundles and the same fraction of the social endowment. In our context, that is, $\mu(E) = \max\{x \in A^*(E) | u_i(x_i) = u_i(t\omega), \forall i \in N, t \in \mathbb{R}_+\}$. However, the classic EE rule is not resource monotonic. Then, the e-EE rule is proposed to overcome this drawback. The e-EE rule fixes an arbitrary reference bundle $e \in \mathring{\mathbb{R}}^l_+$, and gives the agents the shares between which and te they feel indifferent, where t is taken as high as possible. Even more generally, fix a strictly increasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}^l_+$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty, \forall k \in L$, and $\varphi(0) = \mathbf{0}$. We can define the φ-EE rule by $\mu(E) = \max\{x \in A^*(E) | u_i(x_i) = u_i(\varphi(t)), \forall i \in N, t \in \mathbb{R}_+\}$. The φ-EE rule makes all agents indifferent between their shares and the same commodity bundle on the reference curve, that is, $\varphi(t^*)$ for some $t^* \in \mathbb{R}_+$. Hence, these shares are "equal" as viewed by society. Note that the e-EE rule is the φ-EE rule with $\varphi(t) = te$.

We check that on the domain of Leontief preferences, the φ-EE rule is a special case of the generalized egalitarian rules. Note that when ω is fixed, the classic EE rule with $\varphi(t) = t\omega$ is also a special case.

[5] The value of W on $\partial\mathbb{R}^l_+ \backslash \{\mathbf{0}\}$ is irrelevant, since $A^*(E) \cap \partial\mathbb{R}^l_+ = \{\mathbf{0}\}$. More rigorously, W represents a benchmark preference on the interior and the origin of the commodity space.

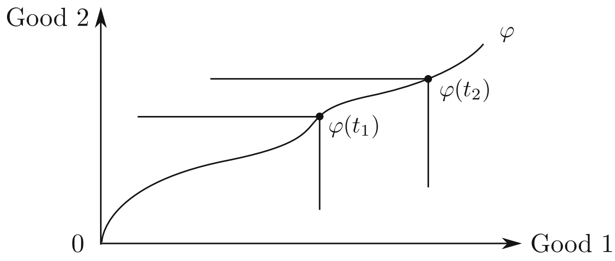

Fig. 2 φ-EE rules

Lemma 3 *Let μ be a φ-EE rule. Define for all $y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$, $W(y) = t$ if and only if $y \in \{\varphi(t)\} - \partial\mathbb{R}^l_+$. Then, $\mu = \mu^W$.*

Proof First, since φ is continuous and strictly increasing, W is well defined, and moreover, continuous and strictly increasing.

Next, fix $E = (N, u_N, \omega)$. Observe that $\forall x \in A^*(E)$ and $\forall i \in N$, $W(x_i) = t$ if and only if $x_i \in \{\varphi(t)\} - \partial\mathbb{R}^l_+$, which is equivalent to $u_i(x_i) = u_i(\varphi(t))$ since $x_i \in \gamma_i$. Hence, $\mu = \mu^W$ by the definitions. □

Figure 2 shows in a two-commodity space, the indifference classes of the benchmark preference W which is defined from φ.

The second interpretation relates to "equal opportunity allocations" proposed by Thomson (1994). Such an allocation is obtained by having each agent choose by himself in *a common choice set*. In this way, it gives the agents equal opportunities. We reformulate the definition in our context.

Let $\mathcal{C}$ be a family of choice sets, where each $C \in \mathcal{C}$ is a nonempty subset of $\mathbb{R}^l_+$.

Definition 4 (Thomson 1994) Given an economy $E = (N, u_N, \omega)$, a feasible allocation x is an equal opportunity allocation relative to the family $\mathcal{C}$ if there exists $C \in \mathcal{C}$ such that $\forall i \in N$, $x_i \in \arg\max_{y \in C} u_i(y)$.

Lemma 4 *Let μ^W be given. Suppose $C(t) = \{y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} | W(y) \leq t\}$ where $t \in \mathbb{R}_+$. Let $\mathcal{C} = \{C(t) | t \in \mathbb{R}_+\}$. Then, $\mu^W(E) = \max\{x \in A^*(E) | x \text{ is an equal opportunity allocation relative to } \mathcal{C}\}$ for all $E \in \mathcal{E}$.*

Proof Let E be given. We only need to show that if $x \in A^*(E)$, then $W(x_i) = W(x_j), \forall i, j \in N$ is equivalent to that x is an equal opportunity allocation relative to the family $\mathcal{C}$. If $W(x_i) = W(x_j), \forall i, j \in N$, then let $t = W(x_i)$, and thus, x_i is the optimal bundle in $C(t)$ for all i since both u_i and W are strictly increasing. Conversely, suppose that x_i is the optimal bundle in $C(t)$ for all i. If WLOG there exist x_1 and x_2, such that $W(x_1) > W(x_2)$, then we must have $t \geq W(x_1) > W(x_2)$. Thus, there must exist $x_2' \in \gamma_2$ such that $x_2' > x_2$ and $W(x_2') < t$. It contradicts that x_2 is the optimal bundle in $C(t)$. □

Hence, a generalized egalitarian rule always picks the Pareto-optimal equal opportunity allocation relative to the family of nested choice sets generated by W. In Example 1, $\mathcal{C}$ is the class of all budget sets with a fixed price. In Example 2, $\mathcal{C}$ is the class of box-shaped sets C with $C = \{y | y \leq \varphi(\lambda)\}$.

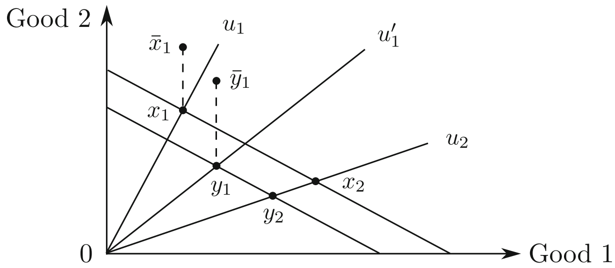

Fig. 3 A counter-example for wasteful allocation

Our first main result is a characterization of generalized egalitarian rules.

Theorem 1 (i) *If a rule μ is in $\mathcal{M}$, then it is efficient, resource monotonic, consistent, group strategy-proof, anonymous and envy-free.*
(ii) *Let a rule μ be efficient, resource monotonic and consistent. If μ is either strategy-proof and anonymous, or envy-free, then $\mu \in \mathcal{M}$.*

In fact, Theorem 1 also holds for a much larger preference domain which is the object of the next section.

The requirement of non-wasteful allocation is very important for Theorem 1. Consider a natural extension of our rules to those which divide up every good. That is, first apply a generalized egalitarian rule μ^W and then allocate the remaining goods equally among the agents. More precisely, this extended rule $\bar{\mu}$ assigns for all $E = (N, \mu_N, \omega)$ and for all $i \in N$, $\bar{\mu}_i(E) = \mu_i^W(E) + \frac{1}{|N|}(\omega - \sum_{i \in N} \mu_i^W(E))$. We show that $\bar{\mu}$ is not SP by a counter-example. For simplicity, suppose that $W = p \cdot x$ where $p > \mathbf{0}$. Let $E = (\{1, 2\}, (u_1, u_2), \omega)$ where (i) $\omega \in \mathbb{R}^2_+$ and ω_2 is large enough so that good 2 is always available in the following discussion; (ii) the slope of the critical set of u_1 is greater than that of u_2. Let u'_1 be such that the slope of its critical set is in between those of u_1 and u_2. See Fig. 3 for an illustration. Let E' be E with u_1 replaced by u'_1. Suppose that $\mu^W(E) = (x_1, x_2)$, $\mu^W(E') = (y_1, y_2)$, $\bar{\mu}(E) = (\bar{x}_1, \bar{x}_2)$ and $\bar{\mu}(E') = (\bar{y}_1, \bar{y}_2)$. Since ω_2 is large enough, then it is always good 1 that is divided up. We check that $y_1^1 > x_1^1$. If $y_1^1 \le x_1^1$, then $W(y_2) = W(y_1) < W(x_1) = W(x_2)$. Hence, $y_2 < x_2$, and thus, $y_1^1 + y_2^1 < x_1^1 + x_2^1 = \omega_1$, which violates the efficiency of μ^W. Once again let ω_2 be large enough such that $\bar{y}_1^2 = y_1^2 + \frac{1}{2}(\omega_2 - y_1^2 - y_2^2) > x_1^2$. Then, after dividing the remaining good 2, $\bar{y}_1 > x_1$, and thus, $u_1(\bar{y}_1) > u_1(x_1) = u_1(\bar{x}_1)$. This example can be easily extended to economies with more goods.

Hence, if one wants a rule to allocate all the goods and be EFFN and SP, then one must carefully design the way that the useless goods are divided. Nicolò (2004) provides such a rule in a two-agent two-good economy. However, there is no result yet in a general economy.

Remark 3 In the characterization of Nicolò (2004), he introduces an incentive compatibility axiom stronger than strategy-proofness — fully implementability in truthful strategies. It requires that a rule is strategy-proof and moreover when a misreport of an agent does not change his own utility, the whole allocation is unaffected. Our rules

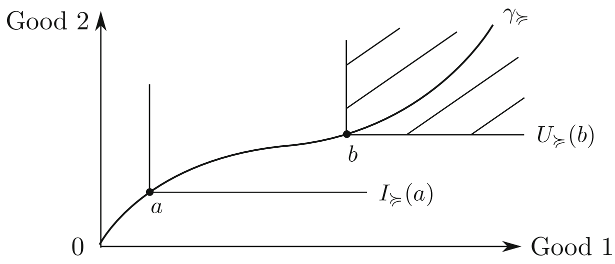

Fig. 4 A generalized Leontief preference in a two-good economy

satisfy this axiom if and only if $\forall x, y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$, $x \geq y$ and $x \neq y$ imply that $W(x) > W(y)$.

4 Generalized Leontief preferences

All the proofs of the results in this section are in the "Appendix".

Let $\succsim$ be a complete and transitive binary relation on $\mathbb{R}^l_+$, $\succ$ and $\sim$ be the corresponding strict and indifferent relations. For all $x \in \mathbb{R}^l_+$, denote by $U_{\succsim}(x) = \{y \in \mathbb{R}^l_+ | y \succsim x\}$ the upper contour set of x, and $I_{\succsim}(x) = \{y \in \mathbb{R}^l_+ | y \sim x\}$ its indifference class.

Definition 5 The set of generalized Leontief preferences is defined by $\mathcal{D} = \{\succsim$ on $\mathbb{R}^l_+ |\succsim$ is continuous and locally non-satiated, and $\forall x \in \mathbb{R}^l_+$, $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ for some $a \in \mathbb{R}^l_+\}$.

Lemma 5 *If* $\succsim \in \mathcal{D}$, *then*

(i) $\succsim$ *is monotone, that is,* $\forall x, y \in \mathbb{R}^l_+$, $x > y$ *implies that* $x \succ y$;
(ii) *for any* $x \in \mathbb{R}^l_+$, $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ *implies that* $I_{\succsim}(x) = \{a\} + \partial\mathbb{R}^l_+$.

Definition 6 For any $\succsim \in \mathcal{D}$, define $\gamma_{\succsim} = \{a \in \mathbb{R}^l_+ : U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ for some $x \in \mathbb{R}^l_+\}$ to be the critical set of the preference $\succsim$.

Clearly, Definition 6 generalizes Definition 1 on the domain of generalized Leontief preferences.

Lemma 6 *For any* $\succsim \in \mathcal{D}$,

(i) $\mathbf{0} \in \gamma_{\succsim}$, *and* $\gamma_{\succsim}$ *is unbounded;*
(ii) *if* $a, b \in \gamma_{\succsim}$ *and* $a \neq b$, *then either* $a < b$ *or* $a > b$, *that is,* $\gamma_{\succsim}$ *is totally ordered;*
(iii) $\gamma_{\succsim}$ *is connected;*
(iv) $\gamma_{\succsim}$ *is closed.*

Figure 4 shows the typical upper contour set, the indifference class and the critical set of a generalized Leontief preference in a two-good economy.

Proposition 1 *For any* $\succsim \in \mathcal{D}$, $\succsim$ *is represented by* $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, *where* $\zeta : \mathbb{R}_+ \to \gamma_{\succsim}$ *is a strictly increasing homeomorphism such that* $\sum_{k \in L} \zeta^k(t) = t$, $\forall t \in \mathbb{R}_+$.

For any $x \in \gamma_{\succsim}$, $x = \zeta(t)$ for some t, and thus, $u(x) = t = \sum_{k \in L} x^k$. Hence, u restricted on $\gamma_{\succsim}$ is a strictly increasing continuous function.

Let $\tilde{\mathcal{U}}$ be the set of all utility functions representing generalized Leontief preferences in the way specified in Proposition 1. Note that $\tilde{\mathcal{U}}$ is a generalization of $\mathcal{U}$, since for any standard Leontief preference represented by $u \in \mathcal{U}$ with $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$, $\zeta(t) = (\lambda_1 t, \ldots, \lambda_l t)$, $\forall t \in \mathbb{R}_+$, and thus, $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, as well.

It is easy to see that under the larger preference domain $\tilde{\mathcal{U}}$, all the previous notions such as economy, rule and generalized egalitarian rule are still well defined. Moreover, as we mentioned before, Theorem 1 still holds when $\mathcal{U}$ is replaced by $\tilde{\mathcal{U}}$.

Let $\tilde{\mathcal{M}}$ denote the class of generalized egalitarian rules under the domain $\tilde{\mathcal{U}}$. For simplicity, we will still use notations such as E, $A^*(E)$ and μ to denote the corresponding notions under the generalized preference domain.

Theorem 2 (i) *If a rule μ is in $\tilde{\mathcal{M}}$, then it is efficient, resource monotonic, consistent, group strategy-proof, anonymous and envy-free.*
(ii) *Let a rule μ be efficient, resource monotonic and consistent. If μ is either strategy-proof and anonymous, or envy-free, then $\mu \in \tilde{\mathcal{M}}$.*

5 The proofs

Generally speaking, the structure of our problem has some resemblance to the "fixed path" methods in the rationing literature, such as the parametric method in Young (1987), and the fixed path rationing method in Moulin (1999). The essential idea of the proof is to investigate how the given axioms impact the range of the rules. We find that the range can be identified with some features which enable us to construct a benchmark preference.

Here we prove Theorem 2. In fact, the result of every step in the following is true under both preference domains. The proofs under $\mathcal{U}$ just involve less cases to check. For the simplicity of presentation, we assume that a rule assigns to every agent an unbounded bundle when the endowment increases, that is, $\forall(N, u_N)$, $\forall i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+) = \{x_i \in \mathbb{R}^l_+ | \mu_i(N, u_N, \omega) = x_i$ for some $\omega \in \mathbb{R}^l_+\}$ is an unbounded subset in $\mathbb{R}^l_+$. This assumption is not necessary. The relaxation of it will be discussed in the "Appendix".

Step 1 If μ is EFFN and RM, then

(i) $\forall(N, u_N)$, $\forall x, x' \in \mu(N, u_N, \mathbb{R}^l_+)$ such that $x \neq x'$, either $x_i < x'_i$, $\forall i \in N$, or $x_i > x'_i$, $\forall i \in N$;
(ii) $\forall(N, u_N)$, $\forall i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+) = \gamma_i$.

Proof Let (N, u_N) be given. Suppose WLOG that $N = \{1, \ldots, n\}$.

(i) Assume that $\mu(\omega) = x$, $\mu(\omega') = x'$, $\omega, \omega' \in \mathbb{R}^l_+$, and $x \neq x'$.
First observe that if $x_j < x'_j$ for some $j \in N$, then $x_i \leq x'_i$ for all $i \in N$. Suppose the contrary WLOG that $x_1 < x'_1$ and $x_2 > x'_2$. Then, $\sum_{i \in N} \min\{x_i, x'_i\} <$

$\sum_{i \in N} x_i \leq \omega$, and $\sum_{i \in N} \min\{x_i, x'_i\} < \sum_{i \in N} x'_i \leq \omega'$. Since μ is RM, then, $\mu_i(\sum_{i \in N} \min\{x_i, x'_i\}) < \min\{x_i, x'_i\}$, $\forall i \in N$, which violates the efficiency of μ.

Next note that if $y \in \mu(\mathbb{R}^l_+)$, then $\mu(\sum_{i \in N} y_i) = y$. Suppose the contrary WLOG that $\mu(\sum_{i \in N} y_i) = y'$ and $y_1 < y'_1$. By our previous result, $y_i \leq y'_i$, $\forall i \in N$. Thus, $\sum_{i \in N} y'_i > \sum_{i \in N} y_i$, which violates feasibility.

Hence, we can take $\omega = \sum_{i \in N} x_i$ and $\omega' = \sum_{i \in N} x'_i$. Suppose WLOG that $x_1 \neq x'_1$. If $x_1 < x'_1$, then we know that $x_i \leq x'_i$, $\forall i \in N$. Thus, $\omega < \omega'$. Since μ is RM, then $x_i < x'_i$, $\forall i \in N$. Similarly, if $x_1 > x'_1$, then $x_i > x'_i$, $\forall i \in N$.

(ii) Suppose the contrary WLOG that $a \in \gamma_1 \setminus \mu_1(\mathbb{R}^l_+)$. Since $\mathbf{0} \in \mu_1(\mathbb{R}^l_+)$ and $\mu_1(\mathbb{R}^l_+)$ is unbounded, then $\underline{\nu} = \{x \in \mu(\mathbb{R}^l_+) | x_1 < a\}$ and $\bar{\nu} = \{x \in \mu(\mathbb{R}^l_+) | x_1 > a\}$ are nonempty. Let $\underline{\omega} = \sup\{\sum_{i \in N} x_i | x \in \underline{\nu}\}$[6] and $\bar{\omega} = \inf\{\sum_{i \in N} x_i | x \in \bar{\nu}\}$. By (i), $\underline{\nu} \cup \bar{\nu} = \mu(\mathbb{R}^l_+)$ is totally ordered, so $\underline{\omega}$ and $\bar{\omega}$ are well defined, and $\underline{\omega} \leq \bar{\omega}$. If $\underline{\omega} < \bar{\omega}$, then pick ω such that $\underline{\omega} < \omega < \bar{\omega}$. By the choice of ω, $\mu(\omega) \notin \underline{\nu} \cup \bar{\nu}$, which is a contradiction. If $\underline{\omega} = \bar{\omega}$, let $y = \sup \underline{\nu} = \inf \bar{\nu}$, and then $y_1 = a$. Let $(y'_i)_{i \in N} = \mu(\sum_{i \in N} y_i)$. By assumption $y'_1 \neq y_1$. If $y_1 < y'_1$, then $y' \in \bar{\nu}$ and thus $y'_i \geq y_i$, $\forall i \in N$. Hence, $\sum_{i \in N} y'_i > \sum_{i \in N} y_i$, which violates the feasibility. If $y_1 > y'_1$, then by a similar argument the efficiency is violated.

□

Step 2 If $\mu \in \tilde{\mathcal{M}}$ is EFFN, RM and CST, then

(i) $\forall (N, u_N)$, $\forall N' \subseteq N$, $(x_i)_{i \in N} \in \mu(N, u_N, \mathbb{R}^l_+)$ implies that $(x_i)_{i \in N'} \in \mu(N', u_{N'}, \mathbb{R}^l_+)$;

(ii) $\forall (N_1, u_{N_1})$ and (N_2, u_{N_2}) such that $N_1 \cap N_2 = \emptyset$, $\forall (x_i)_{i \in N_1} \in \mu(N_1, u_{N_1}, \mathbb{R}^l_+)$ and $(x_i)_{i \in N_2} \in \mu(N_2, u_{N_2}, \mathbb{R}^l_+)$, if for some $i_1 \in N_1$ and $i_2 \in N_2$, $(x_{i_1}, x_{i_2}) \in \mu(\{i_1, i_2\}, (u_{i_1}, u_{i_2}), \mathbb{R}^l_+)$, then $(x_i)_{i \in N} \in \mu(N, u_N, \mathbb{R}^l_+)$ where $N = N_1 \cup N_2$ and $u_N = (u_{N_1}, u_{N_2})$.

Proof Obviously, (i) follows from Step 1 (i) and the definition of consistency.

For (ii), suppose the contrary that under the required condition, $(x_i)_{i \in N} \notin \mu(N, u_N, \mathbb{R}^l_+)$. Then, assume that $\mu(N, u_N, \sum_{i \in N} x_i) = (x'_i)_{i \in N} \neq (x_i)_{i \in N}$. Thus, there must exist some $j \in N$ such that $x'_j < x_j$. Suppose WLOG that $j \in N_1$. By (i), we know that $(x'_i)_{i \in N_1} \in \mu(N_1, u_{N_1}, \mathbb{R}^l_+)$, $(x'_{i_1}, x'_{i_2}) \in \mu(\{i_1, i_2\}, (u_{i_1}, u_{i_2}), \mathbb{R}^l_+)$, and $(x'_i)_{i \in N_2} \in \mu(N_2, u_{N_2}, \mathbb{R}^l_+)$. From our assumption and Step 1, we have that $x'_i < x_i$, $\forall i \in N_1$, and thus $x'_{i_2} < x_{i_2}$, and finally $x'_i < x_i$, $\forall i \in N_2$. Hence, $\sum_{i \in N} x'_i < \sum_{i \in N} x_i$, which violates that μ is EFFN. □

Remark 4 It can also be shown that if μ is EFFN and RM, then both (i) and (ii) of Step 2 are sufficient conditions for μ to be CST.

Step 3 Suppose that μ is EFFN, RM and CST. Then, μ is SP if and only if $\forall (N, u_N)$ such that $|N| = 2$, $\forall i \in N$, $\forall u'_i \in \tilde{\mathcal{U}}$, if $(x_i, x_{-i}) \in \mu(N, u_N, \mathbb{R}^l_+)$ and $(x'_i, x_{-i}) \in \mu(N, u'_N, \mathbb{R}^l_+)$ where $u'_N = (u'_i, u_{-i})$, then $x_i \not< x'_i$.

[6] For all $A \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, $(\sup A)_k = \sup\{a_k : a \in A\}$, $k = 1, \ldots, m$; $\inf A$ is similarly defined.

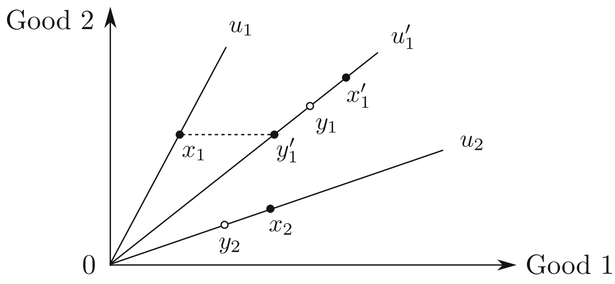

Fig. 5 Necessity for Step 3 (strategy-proofness)

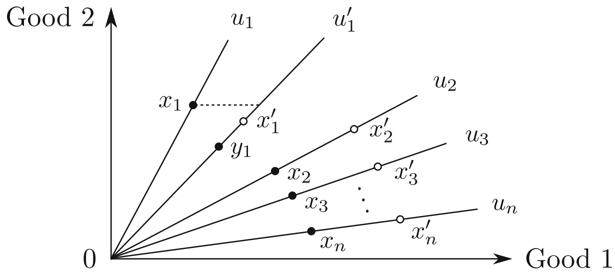

Fig. 6 Sufficiency for Step 3 (strategy-proofness)

Proof For necessity, suppose the contrary WLOG that $N = \{1, 2\}$, and under the required condition, $x_1 < x'_1$. Since γ'_1 is connected, we can find $y'_1 \in \gamma'_1$ such that $y'_1 \in \{x_1\} + \partial\mathbb{R}^l_+$. See Fig. 5 for an illustration in a two-good economy.

Let $\omega = y'_1 + x_2$. By Step 1 (i), $\mu(N, u_N, \omega) = (x_1, x_2)$. We assume that $\mu(N, u'_N, \omega) = (y_1, y_2)$. Since $(x'_1, x_2) \in \mu(N, u'_N, \mathbb{R}^l_+)$ and $x'_1 + x_2 > x_1 + x_2$, then $y_1 < x'_1$ and $y_2 < x_2$. Thus, by efficiency, $y_1 > y'_1 \geq x_1$. This means that in the economy (N, u_N, ω), agent 1 has incentive to misreport his preference, which violates that μ is SP.

For sufficiency, given the required assumption, we want to show that μ is SP. WLOG let (N, u_N, ω) where $N = \{1, \ldots, n\}$, and $u'_1 \in \tilde{\mathcal{U}}$ be given. Let $\mu(N, u_N, \omega) = (x_i)_{i \in N}$, and $\mu(N, u'_N, \omega) = (x'_i)_{i \in N}$ where $u'_N = (u'_1, u_{-1})$. See Fig. 6.

We can find $y_1 \in \gamma'_1$ such that $(y_1, x_2) \in \mu(\{1, 2\}, (u'_1, u_2), \mathbb{R}^l_+)$. By consistency, $(x_1, x_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. By the required assumption, $x_1 \not< y_1$. Hence, if $x'_1 \leq y_1$, then $x_1 \not< x'_1$. Consider the other case that $x'_1 > y_1$. By consistency, $(x_2, \ldots, x_n) \in \mu(N \setminus \{1\}, u_{N \setminus \{1\}}, \mathbb{R}^l_+)$. From Step 2, consider $(N_1, u_{N_1}) = (\{1\}, u'_1)$, $(N_2, u_{N_2}) = (N \setminus \{1\}, u_{N \setminus \{1\}})$, and thus $(y_1, x_2, \ldots, x_n) \in \mu(N, u'_N, \mathbb{R}^l_+)$. Hence, $x'_i > x_i$, $\forall i = 2, \ldots, n$. If $x'_1 > x_1$, then $\omega \geq \sum_{i \in N} x'_i > \sum_{i \in N} x_i$, which violates the efficiency. Hence, $x_1 \not< x'_1$ and agent 1 has no incentive to misreport his preference. □

Step 4 A rule $\mu \in \tilde{\mathcal{M}}$ if and only if μ is EFFN, RM, CST, SP and ANON.

Proof For necessity, let $\mu \in \tilde{\mathcal{M}}$ and (N, u_N, ω) be given. To check efficiency, by Lemma 1, we only need to check that some commodity is divided up. Suppose the

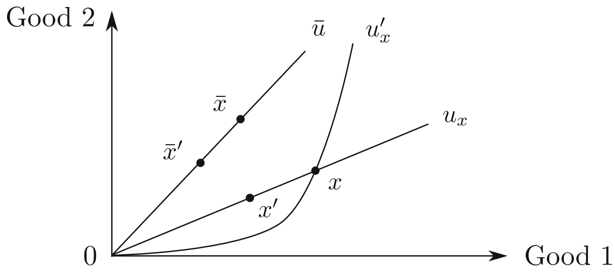

Fig. 7 Independence of the choice of u_x

contrary that $\mu(N, u_N, \omega) = x$ and $\sum_{i\in N} x_i < \omega$. We can find for each $i \in N x'_i \in \gamma_i$ such that $x'_i > x_i$ and $\sum_{i\in N} x'_i \leq \omega$, since γ_i's are connected. Pick $t \in \mathbb{R}_+$ such that $W(x_i) < t < W(x'_i)$, $\forall i \in N$. Since W is continuous and γ_i's are connected, then $W(\gamma_i)$'s are connected. Thus, for each $i \in N$ there exists $y_i \in \gamma_i$ such that $W(y_i) = t$. Clearly, $\sum_{i\in N} y_i < \sum_{i\in N} x'_i \leq \omega$, which contradicts that $\mu(N, u_N, \omega) = x$ by the definition of μ.

To verify that μ is RM, fix ω' such that $\omega' > \omega$. Then, using the similar argument as above, we can show that the bundle allocated to every agent is strictly increased.

Consistency follows from the definition of μ, the efficiency of μ and the assumption that W is strictly increasing.

Strategy-proofness follows from Step 3 and strict increasingness of W.

Lastly, anonymity is simply because μ does not depend on agents' names, but their preferences.

For sufficiency, suppose that μ is EFFN, RM, CST, SP and ANON. Fix $\bar{u} \in \tilde{\mathcal{U}}$. Define $W : \mathring{\mathbf{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ as follows. For any $x \in \mathring{\mathbf{R}}^l_+ \cup \{\mathbf{0}\}$, choose $u_x \in \tilde{\mathcal{U}}$ such that its critical set γ_x contains x. Choose $N = \{1, 2\}$, $u_1 = \bar{u}$, and $u_2 = u_x$. From Step 1, we know that there uniquely exists $\bar{x} \in \bar{\gamma}$ such that $(\bar{x}, x) \in \mu(N, u_N, \mathbb{R}^l_+)$. Define $W(x) = \bar{u}(\bar{x})$. The choice of u_x does not matter, since for any other $u'_x \in \tilde{\mathcal{U}}$ such that $x \in \gamma'_x$ and the corresponding $\bar{x}' \neq \bar{x}$, WLOG say $\bar{x}' < \bar{x}$, then there must be an $x' \in \gamma_x$ such that $x' < x$ and $(\bar{x}', x') \in \mu(N, (\bar{u}, u_x), \mathbb{R}^l_+)$, which contradicts that μ is SP by Step 3. See Fig. 7 for an illustration in a two-good economy. Hence, W is well defined. Note that for any $x \in \bar{\gamma}$, we can pick $u_x = \bar{u}$. Since μ is ANON, then $\mu_i(N, u_N, 2x) = x$, $i = 1, 2$, and thus $W(x) = \bar{u}(x)$ for all $x \in \bar{\gamma}$.

To check that W is strictly increasing, let $x, y \in \mathbb{R}^l_+$ such that $x < y$. We can find $u \in \tilde{\mathcal{U}}$ whose critical set contains both x and y. Find $\bar{x}, \bar{y} \in \bar{\gamma}$ such that $(x, \bar{x}), (y, \bar{y}) \in \mu(\{1, 2\}, (\bar{u}, u), \mathbb{R}^l_+)$. Clearly, $\bar{x} < \bar{y}$, and thus $W(x) = \bar{u}(\bar{x}) < \bar{u}(\bar{y}) = W(y)$.

To verify that W is continuous, we only need to check that $W^{-1}((t, \infty))$ and $W^{-1}([0, s))$ are open sets in $\mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$ when $t \geq 0$ and $s > 0$. Let $t \geq 0$ and $x \in W^{-1}((t, \infty))$ be given. Let u_x and $\bar{x}$ be correspondingly given. By Proposition 1, we can find $\bar{x}_t \in \bar{\gamma}$ such that $\bar{u}(\bar{x}_t) = t$. By Step 1 (ii), there exists $x_t \in \gamma_x$ such that $W(x_t) = t$. See Fig. 8. Since $x \in \gamma_x$ and $W(x) > t$, then $x > x_t$. Thus, there exists $\epsilon > 0$ such that $B_\epsilon(x) = \{y \in \mathbb{R}^l_+ | \, ||y - x|| < \epsilon\} \subseteq \{x_t\} + \mathring{\mathbb{R}}^l_+$. For all $y \in B_\epsilon(x)$, $y > x_t$, and thus $W(y) > W(x_t) = t$. Hence, $B_\epsilon(x) \subseteq W^{-1}((t, \infty))$, which implies that $W^{-1}((t, \infty))$ is open. Similarly, we have that $W^{-1}([0, s))$ is open for all $s > 0$.

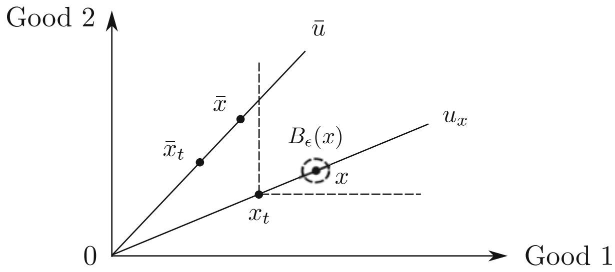

Fig. 8 The continuity of W

Finally, we check that $\forall E = (N, u_N, \omega)$, $\mu(E) = \max\{x \in A^*(E) | W(x_i) = W(x_j),\ \forall i, j \in N\}$. Suppose that $\mu(E) = (x_i^*)_{i \in N}$. Fix $i, j \in N$, and $i \neq j$. Assume WLOG that $1 \notin N$. By the construction of W and the anonymity of μ, there exists $\bar{x}$ such that $(\bar{x}, x_i^*) \in \mu(\{1, i\}, (\bar{u}, u_i), \mathbb{R}_+^l)$. Since μ is CST, $(x_i^*, x_j^*) \in \mu(\{i, j\}, (u_i, u_j), \mathbb{R}_+^l)$. Using Step 2 (ii), consider $N_1 = \{1\}$, $N_2 = \{i, j\}$, we get that $(\bar{x}, x_i^*, x_j^*) \in \mu(\{1, i, j\}, (\bar{u}, u_i, u_j), \mathbb{R}_+^l)$. By the consistency of μ, $(\bar{x}, x_j^*) \in \mu(\{1, j\}, (\bar{u}, u_j), \mathbb{R}_+^l)$. Since μ is ANON, $W(x_i^*) = W(x_j^*)$. Since μ is EFFN, $(x_i^*)_{i \in N} = \max\{x \in A^*(E) | W(x_i) = W(x_j),\ \forall i, j \in N\}$. □

Step 5 If μ is in $\tilde{\mathcal{M}}$, then μ is GSP.

Proof Let (N, u_N, ω), $S \subseteq N$, and $u'_N = (u'_S, u_{-S})$ where $u'_S \in \tilde{\mathcal{U}}_S$ be given. Assume that $\mu(N, u_N, \omega) = x$ and $\mu(N, u'_N, \omega) = x'$. Suppose the contrary that $\forall i \in S$, $u_i(x'_i) \geq u_i(x_i)$, and $\exists j \in S$ such that $u_j(x'_j) > u_j(x_j)$. Hence, $\forall i \in S$, $x'_i \geq x_i$ and $x'_j > x_j$. Thus, $W(x'_j) > W(x_j)$, which by the definition of μ implies that $\forall i \in N \setminus S$, $x'_i > x_i$. Therefore, $\sum_{i \in N} x_i < \sum_{i \in N} x'_i \leq \omega$, which contradicts the efficiency of μ. □

Step 6 A rule μ is in $\tilde{\mathcal{M}}$ if and only if μ is EFFN, RM, CST and EF.

Proof For necessity, let $\mu \in \tilde{\mathcal{M}}$ be given. We only need to check that μ is EF. This simply follows from the definition of μ and the assumption that W is strictly increasing.

For sufficiency, suppose that μ is EFFN, RM, CST and EF. First we show that μ is ANON. Let a bijection π on $\mathbb{N}$, and an economy $E = (N, u_N, \omega)$ be given. Let $E' = (\pi(N), (u_{\pi(i)})_{\pi(N)}, \omega)$ where $u_i = u_{\pi(i)}$, $\forall i \in N$. Assume that $\mu(E) = x$ and $\mu(E') = x'$. Suppose the contrary WLOG that $1, 2 \in N$ and $x_1 < x'_{\pi(1)}$ and $x_2 > x'_{\pi(2)}$. We can find $x'_1 \in \gamma_1$ such that $(x'_1, x'_{\pi(2)}) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}_+^l)$. Note that $x'_1 < x_1 < x'_{\pi(1)}$ since $x'_{\pi(2)} < x_2$. See Fig. 9.

Suppose that $\{1, 2\} \cap \{\pi(1), \pi(2)\} = \emptyset$. Since μ is EFFN and EF, then $\mu(\{2, \pi(2)\}, (u_2, u_2), 2x'_{\pi(2)}) = (x'_{\pi(2)}, x'_{\pi(2)})$. Thus, by Step 2 (ii), $(x'_1, x'_{\pi(2)}, x'_{\pi(2)}, x'_{\pi(1)}) \in \mu(\{1, 2, \pi(2), \pi(1)\}, (u_1, u_2, u_2, u_1), \mathbb{R}_+^l)$, and agent 1 will envy agent $\pi(1)$ which is a contradiction. If $\{1, 2\} \cap \{\pi(1), \pi(2)\} \neq \emptyset$, then pick $i_1, i_2 \in \mathbb{N}$ such that $\{1, 2, \pi(1), \pi(2)\} \cap \{i_1, i_2\} = \emptyset$. From the above result, we know that $(x'_{\pi(1)}, x'_{\pi(2)}) \in \mu(\{i_1, i_2\}, (u_1, u_2), \mathbb{R}_+^l)$. Applying the same argument to the agents

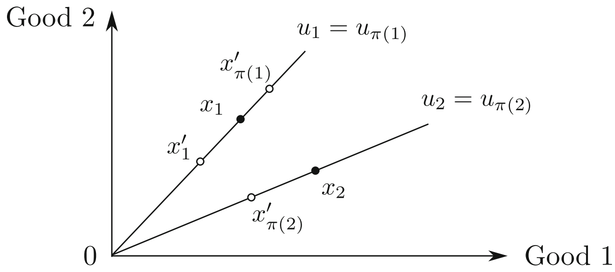

Fig. 9 The anonymity of μ

$1, 2, i_1, i_2$ with the preferences u_1, u_2, u_1, u_2, respectively, we again will get a contradiction.

Now we only need to show that μ is SP. By Step 3, suppose WLOG that $\mu(\{1, 2\}, (u_1, u_2), \omega) = (x_1, x_2)$ and $\mu(\{1, 2\}, (u'_1, u_2), \omega') = (x'_1, x_2)$, and we want to check whether $x_1 \not< x'_1$. Let $u_3 = u'_1$. Since μ is ANON, then $\mu(\{3, 2\}, (u_3, u_2), \omega) = (x'_1, x_2)$. Since μ is CST, then by Step 2 $\mu(\{1, 2, 3\}, (u_1, u_2, u_3), \omega'') = (x_1, x_2, x'_1)$ for some $\omega'' \in \mathbb{R}^l_+$. Since μ is EF, then $u_1(x_1) \geq u_1(x'_1)$, and thus $x_1 \not< x'_1$. □

6 Tightness of the characterization

By Theorem 2, a rule is in $\tilde{\mathcal{M}}$ if and only if one of the following equivalent conditions holds:

(i) it is EFFN, RM, CST, ANON and SP;
(ii) it is EFFN, RM, CST, ANON and GSP;
(iii) it is EFFN, RM, CST and EF.

Our characterization is tight with respect to all these axioms when there are at least two goods in the economy.[7] The tightness result for Theorem 1 is the same.

Drop the efficiency, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \max\{x \in A^*(E)|W(x_i) = t, \forall i \in N; \sum_{i \in N} x_i \leq \omega - te\}$ where W is as in Example 1, and e is the unit vector in the commodity space. It can be checked that $\bar{\mu}$ is well defined, and is RM, CST, ANON, GSP and EF. The key fact used to verify these properties is that if $W(\bar{\mu}_i(E)) = t$, $\forall i \in N$, then $\sum_{i \in N} \bar{\mu}^k_i(E) = \omega^k - t$ for some $k \in L$. However, the allocation given by this rule is never efficient when $\omega > \mathbf{0}$.

Drop the resource monotonicity, and the following rule $\bar{\mu}$ is EFFN, CST, ANON, GSP and EF. Here we define $\bar{\mu}$ in a two-good economy for simplicity, and it can be easily extended to the economies with more than two goods. Consider for each $t \in \mathbb{R}_+$, a parameterized indifference curve $q(t)$ such that: $q(t) = \{x \in \mathbb{R}^2_+|x^1 + x^2 = t\}$ when $t \in [0, 2]$; $q(t) = \{x \in \mathbb{R}^2_+|x^1 + (t-1)x^2 = t$, where $x^1 \geq 1$ or $(t-1)x^1 + x^2 = t$, where $x^1 \leq 1\}$ when $t \in [2, 4]$, and $q(t) = \{x \in \mathbb{R}^2_+|x^1 + 3x^2 = t$, where $x^1 \geq \frac{t}{4}$

[7] It is easy to see that if there is only one good in the economy, then efficiency and either anonymity or envy-freeness will suffice to characterize the rules tightly.

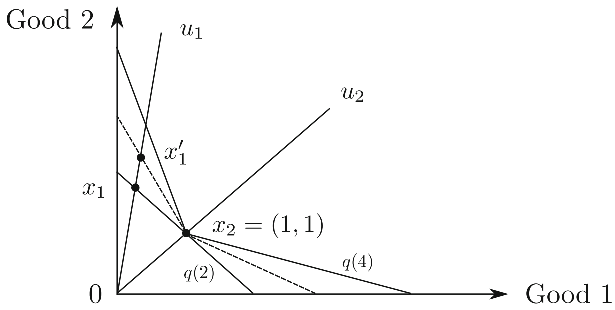

Fig. 10 Tightness of resource monotonicity

or $3x^1 + x^2 = t$, where $x^1 \leq \frac{t}{4}\}$ when $t \in [4, +\infty)$. Let $W : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a set-valued correspondence such that $W(x) = \{t | x \in q(t)\}$. Notice that $W(x)$ is always single-valued except for $W(1, 1) = [2, 4]$. For each $E = (N, u_N, \omega)$, let $\bar{\mu}(E) = \max\{x \in A^*(E) | \prod_{i \in N} I_{W(x_i)}(t) \neq 0$ for some $t \in \mathbb{R}\}$ where $I_{W(x_i)}(t) = 1$ when $t \in W(x_i)$, and $I_{W(x_i)}(t) = 0$ when $t \in \mathbb{R} \setminus W(x_i)$ for all i. It is a well-defined rule, satisfies all the axioms except for resource monotonicity. For a counter-example, consider a preference profile (u_1, u_2) such that γ_2 contains $x_2 = (1, 1)$, as shown in Fig. 10. Let $x_1 \in \gamma_1 \cap q(2)$ and $x_1' \in \gamma_1 \cap q(t)$ for some $t \in (2, 4)$. Then, $\bar{\mu}(x_1 + x_2) = (x_1, x_2)$, $\bar{\mu}(x_1' + x_2) = (x_1', x_2)$. In this case, $x_1 + x_2 < x_1' + x_2$ but agent 2 is not better off. Note that this rule still satisfies the second version of resource monotonicity.

Drop the consistency, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \mu^{W_1}(E)$ if $|N|$ is even and $\bar{\mu}(E) = \mu^{W_2}(E)$ if $|N|$ is odd where W_1 and W_2 are as in Example 1 with different p's. Obviously, $\bar{\mu}$ is EFFN, RM, ANON, GSP and EF, but not CST.

Drop the anonymity, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$ with $1 \notin N$, $\bar{\mu}(E) = \mu^W(E)$ where W is as in Example 1, and for all $E = (N, u_N, \omega)$ with $1 \in N$, $\bar{\mu}(E) = \max\{x \in A^*(E) | 2W(x_1) = W(x_i),\ \forall i \in N \setminus \{1\}\}$. It is a well-defined rule, and is EFFN, RM, CST, GSP, but not ANON. We will prove this result for a general class of such rules in the next section.

Drop the strategy-proofness (and thus the group strategy-proofness), and consider the following rule $\bar{\mu}$ which is EFFN, ANON, RM and CST. Let $\bar{u} \in \tilde{\mathcal{U}}$ be fixed. For all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \mu^W(E)$ if $\forall i \in N$, $u_i \neq \bar{u}$, and if $S = \{j \in N | u_j = \bar{u}\} \neq \emptyset$, $\bar{\mu}(E) = \max\{x \in A^*(E) | 2W(x_j) = W(x_i),\ j \in S,\ i \in N \setminus S\}$ where W is as in Example 1. It is easy to check that $\bar{\mu}$ is well defined and satisfies the above axioms. Figure 11 illustrates that $\bar{\mu}$ is not SP (and thus not GSP) in a two-commodity space. Consider a two-agent economy where their utility profile is as given in Fig. 11. Suppose that $\bar{\mu}(\{1, 2\}, (u_1, u_2), \omega) = (x_1, x_2)$ for some $\omega \in \mathbb{R}^2_+$. Then, agent 1 prefers to report u_1' which is very "close" to u_1. The point on γ_1' "moves" faster than on γ_1, so after agent 1's misreport, it must be that his allocated bundle $x_1' > x_1$.

Drop the envy-freeness, and the above two rules also work as counter-examples. This is because envy-freeness implies anonymity and strategy-proofness when a rule is EFFN, RM and CST.

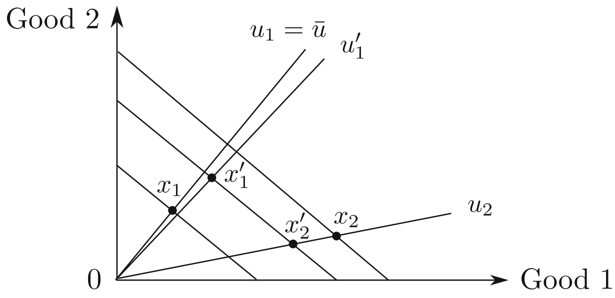

Fig. 11 Tightness of strategy-proofness

7 Agent-specific egalitarian rules

Now we consider a natural extension of $\tilde{\mathcal{M}}$ to a class of non-anonymous rules. While generalized egalitarian rules equalize the agents' final welfare levels according to a benchmark preference over the commodity space, society may measure the welfare of each agent differently. It may attach to each agent i a utility function W_i and equalize the agents' final welfare according to these agent-specific utility functions.

Formally, for all $i \in \mathbb{N}$, let $W_i : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ be a strictly increasing continuous function such that $W_i(\mathbf{0}) = 0$. Let $\mathcal{W}^a = \{W_i | i \in \mathbb{N}\}$ be a set of all agents' welfare indices.

Definition 7 A rule μ is called an agent-specific egalitarian rule if there exists $\mathcal{W}^a$ such that for all $E \in \mathcal{E}$,

$$\mu(E) = \max\{x \in A^*(E) | W_i(x_i) = W_j(x_j), \ \forall i, j \in N\}$$

where $W_i \in \mathcal{W}^a$, $\forall i \in N$. Let $\mathcal{M}^a$ denote the class of agent-specific egalitarian rules.

Using the similar argument as in the proof of Lemma 2, it is easy to see that the analogous result holds, and $\mathcal{M}^a$ is well defined.

Theorem 3 *If μ is in $\mathcal{M}^a$, then μ is efficient, resource monotonic, consistent and group strategy-proof.*

Proof The proof is almost the same as what we did for generalized egalitarian rules. Just by replacing $W(x_i)$ with $W_i(x_i)$ in Step 4 and 5 of Sect. 5, we can get the desired results. □

8 Endowment-specific egalitarian rules and private property

Another extension of $\tilde{\mathcal{M}}$ is natural when we drop the common property assumption. We first introduce the model where every agent has a private endowment. For notational simplicity, we will abuse the previous symbols again to denote the corresponding notions in the model with private property.

An economy E is a triple (N, u_N, ω_N) where $N \subseteq \mathbb{N}$ is a nonempty finite set of agents, $u_N = (u_i)_{i \in N}$ with $u_i \in \tilde{\mathcal{U}}$, $\forall i \in N$, is a preference profile, and $\omega_N = (\omega_i)_{i \in N}$ with $\omega_i \in \mathbb{R}^l_+$, $\forall i \in N$, denotes a vector of private endowments of the agents. Let $\mathcal{E}$ be the set of all economies.

Given (N, ω_N), the set of all feasible allocations is $A(N, \omega_N) = \{x \in \mathbb{R}^{|N| \times l}_+ | \sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i\}$. For any economy $E = (N, u_N, \omega_N)$, the set of non-wasteful feasible allocations is $A^*(E) = A(N, \omega_N) \cap \prod_{i \in N} \gamma_i$ where γ_i is the critical set of u_i. Let $\mathcal{A}^* = \{A^*(E) | E \in \mathcal{E}\}$. A rule is a mapping $\mu : \mathcal{E} \to \mathcal{A}^*$ such that $\mu(E) \in A^*(E)$ for all $E \in \mathcal{E}$.

When the private property is introduced, an important problem is whether the agents are willing to put their own endowments together and participate in the social reallocation. Hence, here we need the individual rationality axiom to guarantee the voluntary participation.

A rule μ is *individually rational* (IR) if $\forall (N, u_N, \omega_N)$, $\forall i \in N$, $u_i(\mu_i(N, u_N, \sum_{i \in N} \omega_i)) \geq u_i(\omega_i)$.

The efficiency, incentive compatibility and fairness axioms are defined in the same way as the previous ones, except a little modification on anonymity and resource monotonicity.

Let π be a bijection on $\mathbb{N}$. A rule μ is *anonymous* if $\forall \pi$, $\forall (N, u_N, \omega_N)$, $\forall i \in N$, $\mu_i(N, u_N, \omega_N) = \mu_{\pi(i)}(\pi(N), (u_{\pi(j)})_{\pi(N)}, (\omega_{\pi(j)})_{\pi(N)})$ where $u_j = u_{\pi(j)}$ and $\omega_j = \omega_{\pi(j)}$, $\forall j \in N$.

A rule μ is *resource monotonic* if $\forall (N, u_N)$, $\forall \omega_N, \omega'_N \in \mathbb{R}^{|N| \times l}_+$, $\omega_i > \omega'_i$ for all $i \in N$ implies that $u_i(\mu_i(\omega_N)) > u_i(\mu_i(\omega'_N))$.

Resource monotonicity is shown to be incompatible with efficiency and individual rationality in Moulin and Thomson (1988). Although they assume a larger preference domain and use another version of resource monotonicity, it is easy to check that with a slight modification their counter-example still works in our context.

Our last result shows that when we allow the welfare index of an agent to depend on his private endowment, we obtain a class of rules which is EFFN, GSP, ANON and IR.

For all $x \in \mathbb{R}^l_+$, let $W_x : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ be a strictly increasing and continuous function such that for all $y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$ with $y \leq x$ and $y^k = x^k$ for some $k \in L$, $W_x(y) = 1$. [8] Let $\mathcal{W}^e = \{W_x | x \in \mathbb{R}^l_+\}$.

Definition 8 A rule μ is called an endowment-specific egalitarian rule, if there exists $\mathcal{W}^e$ such that for all $E \in \mathcal{E}$,

$$\mu(E) = \max\{x \in A^*(E) | W_{\omega_i}(x_i) = W_{\omega_j}(x_j), \ \forall i, j \in N\}$$

where $W_{\omega_i} \in \mathcal{W}^e$, $\forall i \in N$. Let $\mathcal{M}^e$ denote the class of endowment-specific egalitarian rules.

By the analogous result of Lemma 2, $\mathcal{M}^e$ is well defined.

[8] Essentially, what we need is that $W_x(y)$ is some constant which is independent of x.

Theorem 4 *If a rule μ is in $\mathcal{M}^e$, then it is efficient, group strategy-proof, anonymous and individually rational.*

Proof The proof of efficiency, group strategy-proofness and anonymity is basically the same as in Step 4 and 5 of Sect. 5.

To see that μ is IR, note that for all $E = (N, u_N, \omega_N)$, there exists the allocation $x \in A^*(E)$ such that $\forall i \in N$, $u_i(x_i) = u_i(\omega_i)$ and $W_{\omega_i}(x_i) = 1$. Hence, $\forall i \in N$, $\mu_i(E) \geq x_i$ and then $u_i(\mu_i(E)) \geq u_i(\omega_i)$. □

9 Concluding remarks

In this paper, we study fair allocation rules on the generalized Leontief preference domain and achieve very positive results. Nevertheless, there are still some immediate open questions. The characterization of the agent-specific and endowment-specific egalitarian rules remains open. Another intriguing question is how we could drop the non-wastefulness assumption of the rules and still get some positive results. We also observe that recently de Castro et al. (2011) find nice properties of consumption allocation in asymmetric information economies under Maximin preferences, which has some structural resemblance to Leontief preferences without uncertainty. We would like to investigate the relationship between the two problems in the future.

10 Appendix

10.1 The proofs of the results in Section 3

Lemma 5 *If $\succsim \in \mathcal{D}$, then*

(i) *$\succsim$ is monotone, that is, $\forall x, y \in \mathbb{R}^l_+$, $x > y$ implies that $x \succ y$;*
(ii) *for any $x \in \mathbb{R}^l_+$, $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$ implies that $I_{\succsim}(x) = \{a\} + \partial\mathbb{R}^l_+$.*

Proof Let $\succsim \in \mathcal{D}$ be given.

(i) Suppose that $x, y \in \mathbb{R}^l_+$ and $x > y$. Since $\succsim$ is locally non-satiated, we can find $y' < x$ such that $y' \succ y$. Let $U_{\succsim}(y') = \{a\} + \mathbb{R}^l_+$, $a \in \mathbb{R}^l_+$. Since $y' \in U_{\succsim}(y')$ and $x > y'$, then $x \geq a$, and thus $x \in U_{\succsim}(y')$. Hence, $x \succsim y' \succ y$.
(ii) Suppose that $x \in \mathbb{R}^l_+$ and $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$. By (i), $\forall y \in \{a\} + \mathring{\mathbb{R}}^l_+$, $y \succ x$. Now let $y \in \{a\} + \partial\mathbb{R}^l_+$. Since $\succsim$ is continuous, if $y \succ x$, then there exists $y' < y$ such that $y' \succ x$, which contradicts that $U_{\succsim}(x) = \{a\} + \mathbb{R}^l_+$. Hence, $I_{\succsim}(x) = \{a\} + \partial\mathbb{R}^l_+$. □

Lemma 6 *For any $\succsim \in \mathcal{D}$,*

(i) *$\boldsymbol{0} \in \gamma_{\succsim}$, and $\gamma_{\succsim}$ is unbounded;*
(ii) *if $a, b \in \gamma_{\succsim}$ and $a \neq b$, then either $a < b$ or $a > b$, that is, $\gamma_{\succsim}$ is totally ordered;*
(iii) *$\gamma_{\succsim}$ is connected;*
(iv) *$\gamma_{\succsim}$ is closed.*

Proof Let $\succcurlyeq \in \mathcal{D}$ be given.

(i) To see $\mathbf{0} \in \gamma_{\succcurlyeq}$, it suffices to show that $U_{\succcurlyeq}(\mathbf{0}) = \{\mathbf{0}\} + \mathbb{R}^l_+$. Suppose the contrary that $U_{\succcurlyeq}(\mathbf{0}) = \{a\} + \mathbb{R}^l_+$ where $a \neq \mathbf{0}$. Then, it implies that $\mathbf{0} \notin U_{\succcurlyeq}(\mathbf{0})$, a contradiction.
For unboundedness, suppose the contrary that there exists $y \in \mathbb{R}^l_+$ such that $\forall a \in \gamma_{\succcurlyeq}$, $a < y$. Suppose $U_{\succcurlyeq}(y) = \{b\} + \mathbb{R}^l_+$. Then, $b \in \gamma_{\succcurlyeq}$ and $I_{\succcurlyeq}(y) = \{b\} + \partial\mathbb{R}^l_+$. Thus, $b \leq y$ and $y^k = b^k$ for some $k \in \{1, \ldots, l\}$, which is a contradiction.

(ii) Let $a, b \in \gamma_{\succcurlyeq}$ and $a \neq b$. Suppose that $U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$ and $U_{\succcurlyeq}(y) = \{b\} + \mathbb{R}^l_+$, $x, y \in \mathbb{R}^l_+$. It is not true that $x \sim y$, otherwise $a = b$. By Lemma 5 (ii), $a \sim x$ and $b \sim y$. If $x \succ y$, then $a \succ y$ and thus $a \in \{b\} + \mathring{\mathbb{R}}^l_+$, which means $a > b$. Similarly, if $y \succ x$, then $a < b$.

(iii) Define $\rho : \gamma_{\succcurlyeq} \to \mathbb{R}_+$ such that $\rho(x) = \sum_{k \in L} x^k$, $\forall x \in \gamma_{\succcurlyeq}$. It suffices to show that ρ is a homeomorphism.
The injectivity of ρ follows from (ii). We first prove that ρ is surjective. Suppose the contrary that there exists $t \in \mathbb{R}_+ \setminus \rho(\gamma_{\succcurlyeq})$. Then, $\gamma_{\succcurlyeq} = \alpha \cup \beta$ where $\alpha = \{a \in \gamma_{\succcurlyeq} | \rho(a) < t\}$ and $\beta = \{b \in \gamma_{\succcurlyeq} | \rho(b) > t\}$. By (i) we know that $\rho(\mathbf{0}) = 0$, and $\sup \rho(\gamma_{\succcurlyeq}) = \infty$. Hence, $\alpha, \beta \neq \emptyset$. Let $\bar{a} = \sup \alpha$ and $\underline{b} = \inf \beta$. Clearly, $\bar{a}, \underline{b} \in \mathbb{R}^l_+$ and $\bar{a} \leq \underline{b}$. If there exists $h \in L$ such that $\bar{a}^h < \underline{b}^h$, then pick $x \in \mathbb{R}^l_+$ such that $\bar{a} < x$ and $x^h < \underline{b}^h$. Suppose $I_{\succcurlyeq}(x) = \{c\} + \partial\mathbb{R}^l_+$. Thus, $c \in \beta$ and $x \geq c$, which contradicts that $x^h < \underline{b}^h$. Hence, $\bar{a} = \underline{b}$. Then, by (ii), $I_{\succcurlyeq}(\bar{a}) = \{\bar{a}\} + \partial\mathbb{R}^l_+$. Thus, either $\bar{a} \in \alpha$ or $\bar{a} \in \beta$. If $\bar{a} \in \alpha$, then $\rho(\bar{a}) < t$. We can choose $b \in \beta$ such that $\rho(b)$ is arbitrarily close to $\rho(\bar{a})$, and this contradicts that $\rho(b) > t$. Similarly, if $\bar{a} \in \beta$, we can also get a contradiction.
Next observe that for any $x, y \in \gamma_{\succcurlyeq}$, $||x - y|| \leq |\rho(x) - \rho(y)| \leq l||x - y||$,[9] since either $x < y$ or $x > y$. Hence, ρ is a continuous open mapping.

(iv) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in $\gamma_{\succcurlyeq}$ such that $\lim_{n\to\infty} a_n = a$. If $a \notin \gamma_{\succcurlyeq}$, then $\gamma_{\succcurlyeq} = [\gamma_{\succcurlyeq} \cap (\{a\} + \mathbb{R}^l_+)] \cup [\gamma_{\succcurlyeq} \cap (\{a\} - \mathbb{R}^l_+)]$, since $\gamma_{\succcurlyeq}$ is totally ordered and a is the limit of a sequence of elements in $\gamma_{\succcurlyeq}$. This contradicts that $\gamma_{\succcurlyeq}$ is connected.

□

Proposition 1 *For any $\succcurlyeq \in \mathcal{D}$, $\succcurlyeq$ is represented by $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, where $\zeta : \mathbb{R}_+ \to \gamma_{\succcurlyeq}$ is a strictly increasing homeomorphism such that $\sum_{k \in L} \zeta^k(t) = t$, $\forall t \in \mathbb{R}_+$.*

Proof Let $\succcurlyeq \in \mathcal{D}$ be given. Suppose that ρ is defined as in the proof of Lemma 6 (iii). Clearly, ρ is strictly increasing since $\gamma_{\succcurlyeq}$ is totally ordered. Let $\zeta = \rho^{-1}$. Hence, all the properties of ζ follows from those of ρ. Since $\zeta(\mathbb{R}_+)$ is unbounded and ζ is continuous, then $\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$ is bounded and closed for any $x \in \mathbb{R}^l_+$, and thus $u : \mathbb{R}^l_+ \to \mathbb{R}_+$ is well defined.

Now we show that u represents $\succcurlyeq$. If $x \sim y$ and $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$, then $u(x) = u(y) = \sum_{k \in L} a^k$, since ζ is strictly increasing. If $x \succ y$, $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$

[9] $||\cdot||$ is the standard Euclidean norm.

and $I_{\succsim}(y) = \{b\} + \partial\mathbb{R}^l_+$, then by Lemmas 6(ii) and 5(i), $a > b$. Thus, $u(x) = \sum_{k\in L} a^k > \sum_{k\in L} b^k = u(y)$. □

10.2 The relaxation of the unbounded allocation assumption

There are several places in the steps of the proofs to be modified when we drop the assumption that $\forall(N, u_N),\ \forall i \in N,\ \mu_i(N, u_N, \mathbb{R}^l_+)$ is unbounded.

Step 1 (ii) Suppose that μ is EFFN and RM. If for (N, u_N) and $i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+)$ is bounded, then $\mu_i(N, u_N, \mathbb{R}^l_+) = \{x_i \in \gamma_i | x_i < x_i^*\}$ for some $x_i^* \in \gamma_i$, and moreover, there exists $j \in N$ such that $\mu_j(N, u_N, \mathbb{R}^l_+) = \gamma_j$.

Proof Let (N, u_N) and $i \in N$ be given. Suppose that $\mu_i(\mathbb{R}^l_+)$ is bounded. Let $x_i^* = \sup \mu(\mathbb{R}^l_+)$. Since γ_i is closed, then $x_i^* \in \gamma_i$. Note that if $x_i \in \mu_i(\mathbb{R}^l_+)$, then $x_i + \epsilon \in \mu_i(\mathbb{R}^l_+)$ for some $\epsilon > \mathbf{0}$, since μ is RM. Hence, $x_i^* \notin \mu_i(\mathbb{R}^l_+)$. Then, using the similar argument as in the proof of Step 1, we get that $\forall x_i \in \gamma_i$ such that $x_i < x_i^*$, $x_i \in \mu_i(\mathbb{R}^l_+)$. If $\forall i \in N,\ \mu_i(\mathbb{R}^l_+)$ is bounded, then pick $\omega \geq \sum_{i\in N} x_i^*$, and thus $\sum_{i\in N} \mu_i(\omega) < \sum_{i\in N} x_i^* \leq \omega$, which contradicts that u is EFFN. □

Step 3 The sufficiency part.

Proof Let all the assumptions as in the sufficiency proof of Step 3 be given. we only need to check the case when there does not exist $y_1 \in \gamma_1'$ such that $(y_1, x_2) \in \mu(\{1, 2\}, (u_1', u_2), \mathbb{R}^l_+)$. Pick $y_1' \in \gamma_1'$ such that $y_1' > x_1$. By the modified Step 1 (ii), we can find $y_2 \in \gamma_2$ such that $(y_1', y_2) \in \mu(\{1, 2\}, (u_1', u_2), \mathbb{R}^l_+)$, and $y_2 < x_2$. Since μ is CST, $(x_1, x_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. Again by the modified Step 1, there exist $y_1 \in \gamma_1$ such that $(y_1, y_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. Since $y_2 < x_2$, then $y_1 < x_1$, and thus $y_1 < y_1'$, which contradicts our assumption. □

Step 4 The sufficiency part.

Proof We first show the following two statements:

(i) If μ is EFFN, RM and ANON, then $\forall(N, u_N)$ such that $\forall i \in N,\ u_i = u$, $\mu_i(N, u_N, \mathbb{R}^l_+) = \gamma_i,\ \forall i \in N$;
(ii) If μ is EFFN, RM, ANON and SP, then $\forall(N, u_N)$ such that $|N| = 2$ and γ_i is unbounded in every commodity for some $i \in N$, $\mu_j(N, u_N, \mathbb{R}^l_+) = \gamma_j$ where $j \in N$ and $j \neq i$.

The result (i) follows from Remark 1 and the modified Step 1.

For (ii), let (N, u_N) which satisfies the required conditions be given. By the modified Step 1, suppose the contrary that $\mu_j(\mathbb{R}^l_+)$ is bounded where $j \in N$ and $j \neq i$. Thus, when ω is big enough, agent j would pretend to have agent i's preference, since his allocation would be unbounded in every dimension by statement (i) and the assumption on γ_i. This contradicts that μ is SP.

Then, the construction of W is basically the same except that $\bar{u}$ should be chosen such that its critical set is unbounded in every dimension. By statement (ii), W is well defined. The rest of the proof is the same. □

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